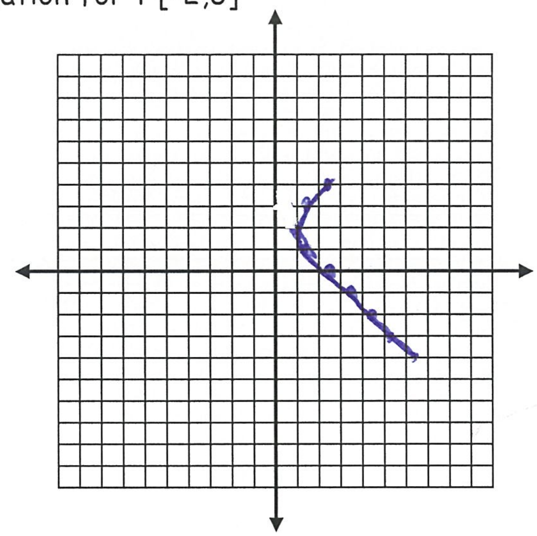
Parametric and Polar Equations Review Name Key

1. Fill in the table and sketch the parametric equation for t [-2,6]

$x = \sqrt{t^2 + 1}$

$y = 2 - t$

t	x	y
-2	$\sqrt{5}$	4
-1	$\sqrt{2}$	3
0	1	2
1	$\sqrt{2}$	1
2	$\sqrt{5}$	0
3	$\sqrt{10}$	-1
4	$\sqrt{17}$	-2
5	$\sqrt{26}$	-3
6	$\sqrt{37}$	-4



Problems 2 – 11: Eliminate the parameter to write the parametric equations as a rectangular equation.

2. $x = \dfrac{1}{t-2}$ $\quad t = \dfrac{1}{x} + 2$

$y = 4t + 5$

$y = 4\left(\dfrac{1}{x} + 2\right) + 5$

$y = \dfrac{4}{x} + 13$

3. $x = 6 - t$ $\quad t = 6 - x$

$y = \sqrt{3t - 4}$

$y = \sqrt{3(6-x) - 4}$

$y = \sqrt{18 - 3x - 4}$

$y = \sqrt{14 - 3x}$

4. $x = \frac{1}{2}t + 4$ $\quad t = 2(x-4)$

$y = t^3$

$y = (2x - 8)^3$

5. $x = 3 \csc t \rightarrow \csc t = \dfrac{x}{3}$

$y = 3 \cot^2 t$ $\quad \cot^2 t = \dfrac{y}{3}$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$1 + \left(\dfrac{y}{3}\right) = \left(\dfrac{x}{3}\right)^2$

$1 = \dfrac{x^2}{9} - \dfrac{y}{3}$

6. $x = 4 \sin(2t)$

$y = 2 \cos(2t)$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\dfrac{x}{4}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$

$\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

7. $x = \cos t$

$y = 2 \sin^2 t$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\dfrac{y}{2}\right) + x^2 = 1$

8. $x = 4 \sec t$

$y = 3 \tan t$

$\tan^2\theta + 1 = \sec^2\theta$

$\left(\dfrac{y}{3}\right)^2 + 1 = \left(\dfrac{x}{4}\right)^2$

$1 = \dfrac{x^2}{16} - \dfrac{y^2}{9}$

9. $x = 4 + 2 \cos t$

$y = -1 + 4 \sin t$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\dfrac{y+1}{4}\right)^2 + \left(\dfrac{x-4}{2}\right)^2 = 1$

$\dfrac{(y+1)^2}{16} + \dfrac{(x-4)^2}{4} = 1$

10. $x = -4 + 3\tan^2 t$

$y = 7 - 2 \sec t$

$\tan^2\theta + 1 = \sec^2\theta$

$\left(\dfrac{x+4}{3}\right) + 1 = \left(\dfrac{y-7}{-2}\right)^2$

$1 = \dfrac{(y-7)^2}{4} - \dfrac{x+4}{3}$

① Pick anything for X to equal

② Plug into y=

Problems 11 and 12: Write two new sets of parametric equations for the following rectangular equations.

11. $y = (x + 2)^3 - 4$

$X = t - 1$ $X = 3t + 2$

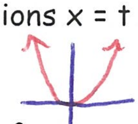
$y = (t+1)^3 - 4$ $y = (3t+4)^3 - 4$

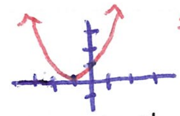
12. $x = \sqrt{y^2 - 3}$

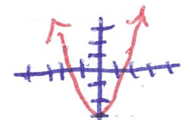
$X = 5t$ $X = t - 2$

$y = \sqrt{(5t)^2 - 3}$ $y = \sqrt{(t-2)^2 - 3}$

13. For the parametric equations $x = t$ and $y = t^2$

a) 

b)  shift left 1 unit

c)  shift down 3 units

a) Sketch the graph.

b) Graph $x = t - 1$ and $y = t^2$. How does this compare to the graph in part (a)?

c) Graph $x = t$ and $y = t^2 - 3$. How does this compare to the graph in part (a)?

d) Write parametric equations which will give the graph in part (a) a vertical stretch by a factor of 2 and move the graph 5 units to the right. (Hint: Verify on your calculator!)

d) $X = t + 5, \quad y = 2t^2$

14. Do the following sets of parametric equations cross at the same time so they collide or do their paths just intersect? Justify your answer.

→ ✱ $x_1 = x_2$, find t, plug into y_1, y_2

a) $x_1 = 3 - t$ and $x_2 = t + 19$

$y_1 = t^2 - 60$ $y_2 = t + 12$

$3 - t = t + 19$ $2t = -16$ $t = -8$

$y_1 = (-8)^2 - 60$ $y_1 = 4$

$y_2 = -8 + 12$ $y_2 = 4$

They collide at $t = -8$, $(11, 4)$

b) $x_1 = 3 - t$ and $x_2 = 3 - 2t$

$y_1 = 2t + 1$ $y_2 = 2 + 3t$

$3 - t = 3 - 2t$ $t = 0$

$y_1 = 2(0) + 1$ $y_1 = 1$ $y_2 = 2$ Do not collide at $t = 0$.

Check $y_1 = y_2$ $2t + 1 = 2 + 3t \to t = -1$ $x_1 = 3 - -1 = 4$ $x_2 = 5$ The objects do not collide

c) $x_1 = 4t$ and $x_2 = 5t - 6$

$y_1 = \frac{1}{2}t + 5$ $y_2 = t + 2$

$4t = 5t - 6$ $t = 6$

$y_1 = 3 + 5$ $y_1 = 8$

$y_2 = 6 + 2$ $y_2 = 8$

The objects collide at $t = 6$, $(24, 8)$

15. Find the values of t that generated the graph described by the parametric equations:

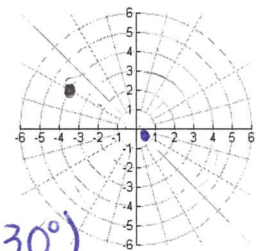
$x = t - 1$ and $y = \frac{1}{2}t + 2$

t	x	y
-4	-5	0
-2	-3	1
0	-1	2
2	1	3
4	3	4

Describe your thought process in solving for t.

set the x-value equal to the $x = t - 1$ equation and solve for t.

Plot the point given in polar coordinates and find two additional polar representations of the point, using -360° < θ < 360°.



6. (4, 150°)

$(-4, -30^\circ)$ $(-4, 330^\circ)$

$(4, -210)$

7. $(-\frac{1}{2}, -210^\circ)$

$(-\frac{1}{2}, 150^\circ)$

$(\frac{1}{2}, 330^\circ)$ $(\frac{1}{2}, -30^\circ)$

Find the corresponding rectangular coordinates for the point given in polar coordinates.

(r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

8. $(5, -\frac{\pi}{6})$

$(5\cos(-\frac{\pi}{6}), 5\sin(-\frac{\pi}{6}))$

$(\frac{5\sqrt{3}}{2}, -\frac{5}{2})$

9. (-2, 135°)

$(-2\cos 135^\circ, -2\sin 135^\circ)$

$(-\sqrt{2}, -\sqrt{2})$

Find the polar coordinates for 0 < θ < 360°. Pay attention to the quadrant!

10. (-4, -4)

$r = \sqrt{(-4)^2 + (-4)^2}$

$\theta = \tan^{-1}(1)$

$\theta = 45^\circ + 180^\circ$

$(4\sqrt{2}, 225^\circ)$ ← * Add 180° or π to make it in third Quadrant

11. $(2, -2\sqrt{3})$

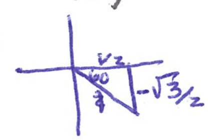
$\theta = \tan^{-1}(-\sqrt{3})$

$r = \sqrt{2^2 + (2\sqrt{3})^2} = 4$ $\theta = -60^\circ$

$(4, -60^\circ)$

$x^2 + y^2 = r^2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$



Convert the rectangular equation to polar form. (solve for r)

12. $x^2 + y^2 - 6y = 0$

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$r^2 - 6r\sin\theta = 0$

$r(r - 6\sin\theta) = 0$

$\cancel{r = 0}$ $\boxed{r = 6\sin\theta}$

13. $5x + 7y = 12$

$5r\cos\theta + 7r\sin\theta = 12$

$r(5\cos\theta + 7\sin\theta) = 12$

$$\boxed{r = \frac{12}{5\cos\theta + 7\sin\theta}}$$

Convert the polar equation to rectangular form.

14. r = 4 sinθ

$y = r\sin\theta$

$\sin\theta = \frac{y}{r}$

$r \cdot r = 4\left(\frac{y}{r}\right) \cdot r$

$r^2 = 4y$

$x^2 + y^2 = 4y$

complete the square!

$x^2 + y^2 - 4y + 4 = 0 + 4$

$x^2 + (y - 2)^2 = 4$

it's a circle!

15. $r = \frac{4}{1 - \cos\theta}$ $(1 - \cos\theta)$ $(1 - \cos)$

$r - r\cos\theta = 4$

$\sqrt{x^2 - y^2} - x = 4$

$\sqrt{x^2 + y^2}^2 = (x + 4)^2$

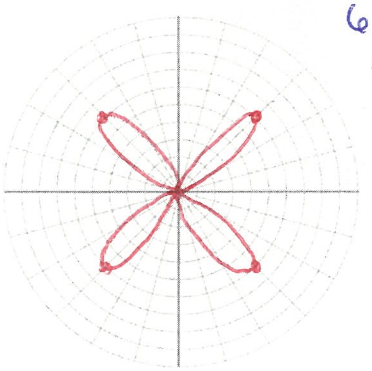
$x^2 + y^2 = x^2 + 8x + 16$

$y^2 = 8x + 16$

Parabola

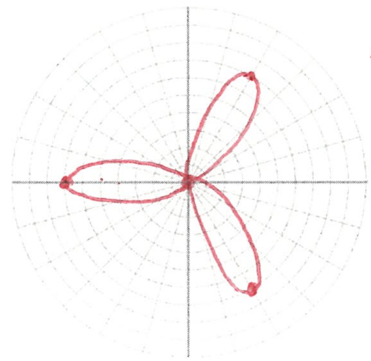
Graph

16. $r = 6\sin 2\theta$



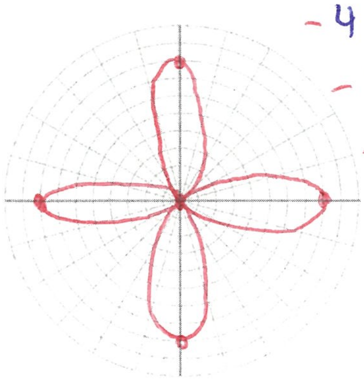
4 petals
6 radius (length
90° apart
1st petal at 45°
because
$\sin 2(45) = \sin 90 = 1$

17. $r = -7\cos 3\theta$



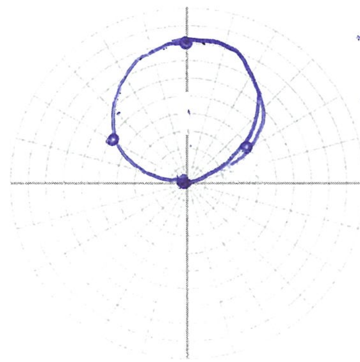
- 3 petals
- 7 radius
- 120° apart
- 1st petal is on the negative x-axis because of -7

14. $r = -8\cos 2\theta$



- 4 petals
- 8 radius
- 90° apart
- 1st petal on negative x-axis

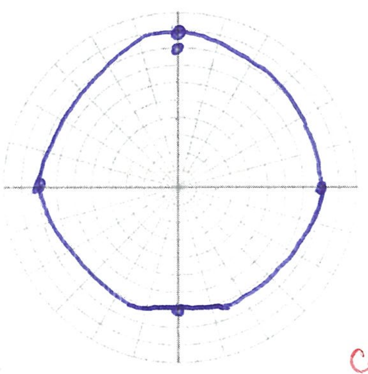
15. $r = 8\sin\theta$



- circle
- on y-axis

$8\sin 0 = 0$
$8\sin 90 = 8$
$8\sin 30 = 4$
$8\sin 150 = 4$

16. $r = 8 + \sin\theta$

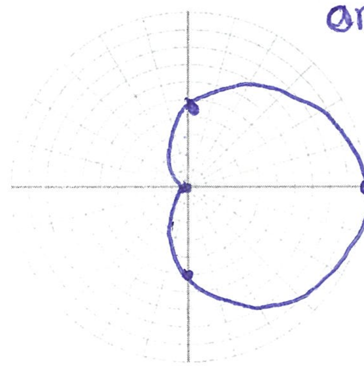


centered around y-axis

$8 + \sin 0 = 8$
$8 + \sin 90 = 9$
$8 + \sin 180 = 8$
$8 + \sin 270 = 7$

Convex limagon

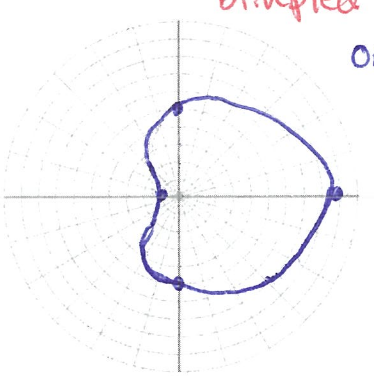
17. $r = 5 + 5\cos\theta$



cardioid
on x-axis

$5 + 5\cos 0 = 10$
$5 + 5\cos 90 = 5$
$5 + 5\cos 180 = 0$
$5 + 5\cos 270 = 5$

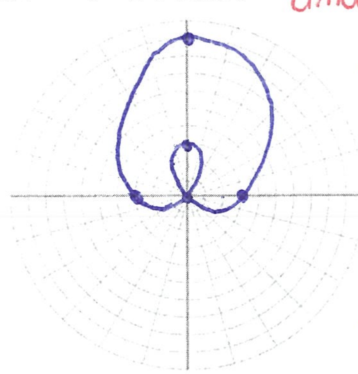
18. $r = 5 + 4\cos\theta$



Dimpled limagon
on x-axis

$5 + 4\cos 0 = 9$
$5 + 4\cos 90 = 5$
$5 + 4\cos 180 = 1$
$5 + 4\cos 270 = 5$

19. $r = 3 + 6\sin\theta$



Limagon w/ loop
on y-axis

$3 + 6\sin 0 = 3$
$3 + 6\sin 90 = 9$
$3 + 6\sin 180 = 3$
$3 + 6\sin 270 = -3$

Remember $\theta = \cos^{-1}(w) \rightarrow 0 \le \theta \le \pi$

$\theta = \sin^{-1}(w)$ and $\theta = \tan^{-1}(w) \rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

TRIGONOMETRY REVIEW

Find the exact value without a calculator.

arccos = $\cos^{-1}$
arcsin = $\sin^{-1}$
arctan = $\tan^{-1}$

21. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$ 22. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ 23. $\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

24. $\cos^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$ 25. $\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$ 26. $\arccos\left(\sin\left(\frac{\pi}{3}\right)\right)$

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

27. $\sin\left(\tan^{-1}\left(\sqrt{3}\right)\right)$ 28. $\cos\left(\tan^{-1}(-1)\right)$ 29. $\tan^{-1}(\cos(\pi))$

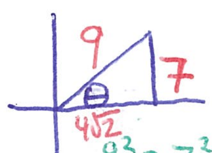
$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\tan^{-1}(-1) = -\frac{\pi}{4}$

30. Find the exact value. Draw a Picture!

a. $cos(arcsin(\frac{7}{9}))$ b. $sin(arctan(\frac{3}{5}))$ c. $\cancel{arcsin}(\cancel{sin}(\frac{2}{3}))$ since $\frac{2}{3}$ radians is in QI

$\cos(\theta)$

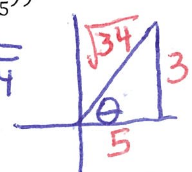
$\boxed{= \frac{4\sqrt{2}}{9}}$



$9^2 = 7^2 + x^2$

$\sin\theta = \frac{3}{\sqrt{34}}$

$\boxed{= \frac{3\sqrt{34}}{34}}$



$c^2 = 5^2 + 3^2$

$\frac{2}{3}$

31. Find all solutions to the following trig equations.

Factor!

$\cos^2 x = 1 - \sin^2 x$

a. $2sinx^{2} - sinx - 2 = 1$ b. $cos^2 x = 1 - sinx$ c. $sinx - 2sinxcosx = 0$

$\sin x(1 - 2\cos x) = 0$

$\sin x = 0$ $\cos x = \frac{1}{2}$

$\boxed{x = 0 + 2\pi n \quad x = \frac{\pi}{3} + 2\pi n}$

COMPLEX NUMBER PRACTICE

32. Write the complex numbers in polar form (trigonometric form)

(a) $z = 2 - 2i$ $2\sqrt{2}(\cos(-45) + i\sin(-45))$

(b) $w = -1 - \sqrt{3}i$ $2(\cos(240) + i\sin(240))$

(c) $y = 4\sqrt{3} + 4i$ $8(\cos(30) + i\sin(30))$

(d) $x = -\sqrt{5} + \sqrt{5}i$ $\sqrt{10}(\cos(135) + i\sin(135))$

31 ⓐ $2\sin^2 x - \sin x - 3 = 0$

$(2\sin x - 3)(\sin x + 1) = 0$

$\cancel{\sin x = \frac{3}{2}}$ $\sin x = -1$

$\boxed{x = \frac{3\pi}{2} + 2\pi n}$

ⓑ $\cancel{1} - \sin^2 x = \cancel{1} - \sin x$

$0 = \sin^2 x - \sin x$

$0 = \sin x(\sin x - 1)$ $\sin x = 1$

$x = 0 + 2\pi n \quad \frac{\pi}{2} + 2\pi n$

$\left(2^{\frac{3}{2}}\right)^7 = 2^{21/2} = 2^{10+1/2} = 2^{10}\sqrt{2}$

33. Using the complex numbers w-z above, simplify the following using polar form.

a. $z \cdot w = 4\sqrt{2}\cos 195° + 4\sqrt{2}\, i \sin 195°$

b. $x \div w = \frac{\sqrt{10}}{2}\left(\cos(-105°) + i\sin(-105°)\right)$

c. $y \cdot x$

d. $z^7 = (2\sqrt{2})^7\left(\cos(-315°) + i\sin(-315°)\right) = 1024\sqrt{2}\left(\cos(-315) + i\sin(-315)\right)$

e. $w^4 = 2^4\left(\cos(4\cdot 240) + i\sin(4\cdot 240)\right) = 16(\cos 960° + i\sin 960°)$

OR $= 16(\cos 240 + i\sin 240)$

★ subtract 360° to get $0 \le \theta \le 360$

c) $y \cdot x = 8\sqrt{10}\left(\cos(165) + i\sin(165)\right)$

34. Write in simplified polar form.

a. $(3+2i)^{30}$

$\left(\sqrt{13}\left(\cos(33.7°) + i\sin(33.7°)\right)\right)^{30}$

$= 13^{15}\left(\cos(1011°) + i\sin(1011°)\right) = \boxed{13^{15}(\cos 291° + i\sin 291°)}$

$\sqrt{13}^{30}$

$13^{\frac{30}{2}}$

13^{15}

b. $(2-6i)^{21}$

$= \left((2\sqrt{10})\left(\cos(-71.6°) + i\sin(-71.6°)\right)\right)^{21}$

$= 2^{21}\cdot 10^{10}\cdot\sqrt{10}\left(\cos(-1503.6) + i\sin(-1503.6)\right)$

$= \boxed{2^{21}\cdot 10^{10}\cdot\sqrt{10}\left(\cos(296.4°) + i\sin(296.4°)\right)}$

35. ECCENTRICITY – Find the eccentricity and identify the conic section

★ Sketch the conic section!

a. $r = \frac{7}{3 - \frac{2}{5}\cos\theta}$ $\frac{1/3}{1/3}$

$r = \frac{7/3}{1 - \frac{2}{15}\cos\theta}$

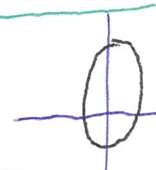
$e < 1$ $\boxed{e = \frac{2}{15}}$ $\boxed{\text{Ellipse}}$

along x-axis because of $\cos\theta$!

b. $r = \frac{4}{4 + \frac{1}{4}\sin\theta}$ $\cdot\frac{1}{4}$

$e = \frac{1}{16}$

Ellipse



along y-axis because of $\sin\theta$

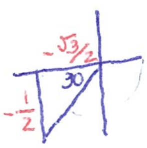
EXTRA PRACTICE WITH POLAR

Convert to rectangular coordinates:

36. $\left(-5, -\frac{5\pi}{6}\right)$

$x = -5\cos\left(-\frac{5\pi}{6}\right)$

$y = -5\sin\left(-\frac{5\pi}{6}\right)$



$\boxed{\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)}$

Convert to polar coordinates:

37. $(-6, 6\sqrt{3})$; $r \le 0$ and $0 \le \theta \le 2\pi$

2nd Quadrant → so add π

$r = \sqrt{(-6)^2 + (6\sqrt{3})^2}$

$r = 12$

$\tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) = -\frac{\pi}{3} + \pi$

$\boxed{\left(12, \frac{2\pi}{3}\right)}$

Change to a rectangular equation:

multiply by r on both sides!

38. $r = -3\cos\theta$

$r^2 = -3r\cos\theta$

$x^2 + y^2 = -3x$

$x^2 + 3x + \frac{9}{4} + y^2 = 0 + \frac{9}{4}$

$\boxed{\left(x + \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}}$

Change to a polar equation:

39. $x + y = 2x$

$r\cos\theta + r\sin\theta = 2r\cos\theta$

$r\sin\theta = r\cos\theta$

$r(\sin\theta - \cos\theta) = 0$

$\cancel{r = 0}$, $\boxed{\sin\theta = \cos\theta}$

Obtain the rectangular equation by eliminating the parameter.

40. $x = 3t - 7$, $y = -6t + 4$

$t = \frac{x+7}{3}$

$y = -6\left(\frac{x+7}{3}\right) + 4$

$\boxed{y = -2x - 10}$

41. $x = -3\cos\theta$, $y = 3\sin^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{y}{3} + \left(\frac{x}{-3}\right)^2 = 1$

$9\left(\frac{y}{3} + \frac{x^2}{9} = 1\right)$

$\boxed{3y + x^2 = 9}$