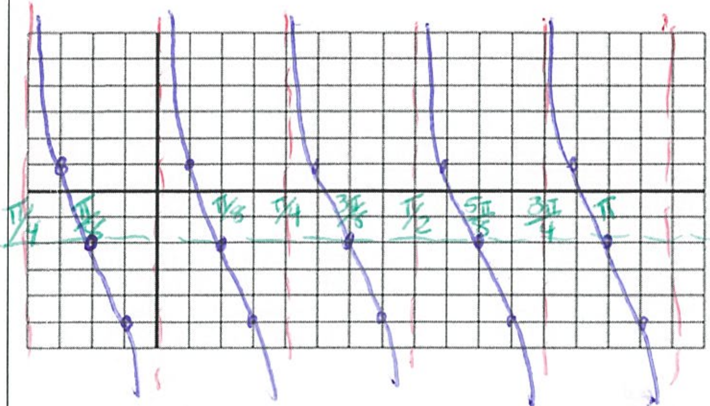
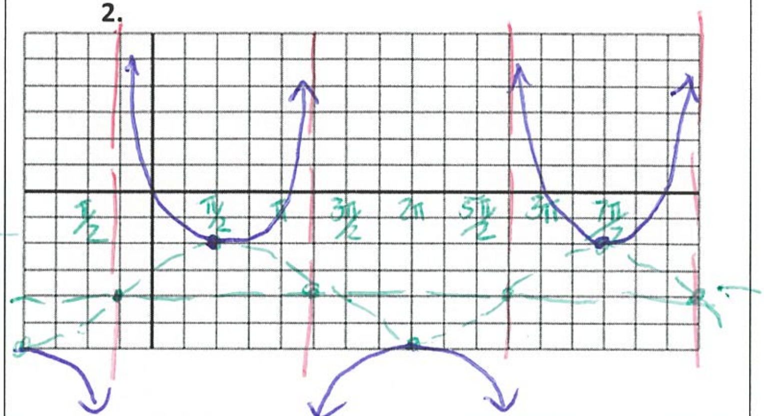
## IM3H More Final Review

Module 4

**1.** $f(x) = 2 - 3\tan 4\left(x + \frac{\pi}{8}\right)$ Period $= \frac{\pi}{4}$



**2.** $y = 2\csc\frac{2}{3}\left(x + \frac{\pi}{4}\right) - 4$ Period: $3\pi$



**3. Find all solutions in the equation in the interval [0, 2π).**

**a.** $\csc^2 x - \csc x - 2 = 0$

$(\csc x - 2)(\csc x + 1) = 0$

$\csc x = 2 \qquad \csc x = -1$

$\sin x = \frac{1}{2} \qquad \sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

**b.** $3 - \sin^2 3x + 2\cos 3x = 5$

$3 - (1 - \cos^2 3x) + 2\cos 3x = 5$

$3 - 1 + \cos^2 3x + 2\cos 3x = 5$

$\cos^2 3x + 2\cos 3x - 3 = 0$

$(\cos 3x + 3)(\cos 3x - 1) = 0$

$\cancel{\cos 3x = -3} \qquad \cos 3x = 1$

$3x = 0 \qquad 3x = \pi$

$x = 0 \qquad x = \pi/3 \qquad x = \frac{\pi}{3} + \frac{2\pi}{3}n$

$x = \pi, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}$

**c.** $\sec x \sin x - 3\sin x = 0$

$\sin x(\sec x - 3) = 0$

$\sin x = 0 \qquad \sec x = 3$

$\boxed{x = 0, \pi} \qquad \cos x = 1/3$

$x =$ calculator

**d.** $3\cot^2 x - 1 = 0$

$\cot x = \pm\sqrt{\frac{1}{3}}$

$\cot x = \pm\frac{1}{\sqrt{3}}$

$\tan x = \pm\sqrt{3}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

**e.** $2\sin^2 3x + 5\sin 3x = 3$

$2\sin^2 3x + 5\sin 3x - 3 = 0$

$(2\sin 3x - 1)(\sin 3x + 3) = 0$

$2\sin 3x - 1 = 0 \qquad \cancel{\sin 3x = -3}$

$\sin 3x = \frac{1}{2}$

$3x = \frac{\pi}{6} \qquad 3x = \frac{5\pi}{6}$

$x = \frac{\pi}{18} \qquad x = \frac{5\pi}{18}$

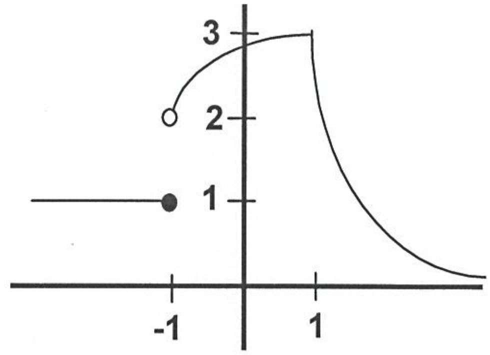
**f.** $2\tan^2\frac{x}{4} - \tan\frac{x}{4} - 6 = 0,$

$(2\tan\frac{x}{4} + 3)(\tan\frac{x}{4} - 2)$

calculator problem

## Module 8

**81.** The picture on the left shows the graph of a certain function.
Based on that graph, answer the questions:




a) $\lim_{x\to -1^-} f(x) = 1$

b) $\lim_{x\to -1^+} f(x) = 2$

c) $\lim_{x\to 1} f(x) = 3$

d) $\lim_{x\to 0} f(x) = 2.8$

e) Is the function continuous at x = -1?
NO

f) Is the function continuous at x = 1?
Yes

g) Is the function differentiable at x = -1?
Skip

h) Is the function differentiable at x = 1?
Skip

i) Is f'(0) positive, negative, or zero?
(slope) Positive

k) What is f'(-2) ?
0 (slope)

Skip g, h

**82.** Find each of the following limits (show your work):

a) $\lim_{x\to 3} 4\pi = 4\pi$

b) $\lim_{x\to 3} \frac{x^2-2x}{x+3} = \frac{1}{2}$

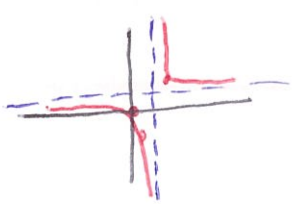
$\frac{9-6}{6}$

c) $\lim_{x\to 3} \frac{3-x}{x^2+2x-15} = \frac{-\cancel{(x-3)}}{(x+5)\cancel{(x-3)}}$

$\frac{0}{9+6-15} = \frac{0}{0}$

$\lim_{x\to 3} \frac{-1}{x+5} = -\frac{1}{8}$

d) $\lim_{x\to 1^+} \frac{x}{x-1} = \infty$



e) $\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$

f) $\lim_{x\to 1} \frac{x}{x-1} = \text{DNE}$

l) $\lim_{x\to -\infty} \frac{3x^2-1}{2-3x-4x^2} = -3/4$

HA ⇒ $y = -3/4$

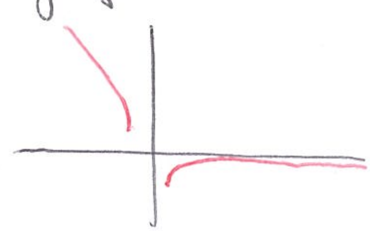
m) $\lim_{x\to -\infty} \frac{3x^2-1}{2-3x} = -\infty$

| | $x$ | $2/3$ | |
|---|---|---|---|
| $-3x$ | $3x^2$ | $-2x$ | $-1/3$ |
| $2$ | $2x$ | $4/3$ | |

Slant: $y = x + 2/3$
Asymptote

n) $\lim_{x\to\infty} \sqrt{x^2-1} - x = 0$

graph



**83.** Consider the following function: $f(x)=\begin{cases} x^2, & \text{if } x\geq 0 \\ x-2, & \text{if } x<0 \end{cases}$

a) Find $\lim_{x\to 0^-} f(x)$ $= -2$

b) Find $\lim_{x\to 0^+} f(x)$ $= 0$

c) Find $\lim_{x\to 2} f(x)$ (note that x approaches *two*, not *zero*) $= 4$

d) Is the function continuous at x = 0, No, $\lim_{x\to 0} f(x) = DNE$

f) Is $f(x)=\begin{cases} \dfrac{x^2-1}{x+1}, & \text{if } x\neq -1 \\ 17 & \text{if } x=-1 \end{cases}$ continuous at -1 ? If not, is the discontinuity removable? NOT continuous

$\frac{(-1)^2-1}{-1+1} = \frac{0}{0}$ $\quad \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = 0$ $\quad$ Yes, it is removable.

g) Is there a value of *k* that makes the function *g* continuous at *x = 0*? If so, what is that value?

$g(x)=\begin{cases} x-2, & \text{if } x\leq 0 \\ k(3-2x) & \text{if } x>0 \end{cases}$

$x-2 = k(3-2x)$ when $x=0$

$-2 = k(3)$

$k = -2/3$

**84.** Find the value of *k*, if any, that would make the following function continuous at ~~x = 4~~. $x=2$

$f(x)=\begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x\neq 2 \\ k & \text{if } x=2 \end{cases}$

$\frac{x^2-4}{x-2} = k$ when $x=2$

$\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = k$

$x+2=k, \boxed{k=4}$

**85.** Use the *definition* of derivative to find the derivative of the function

a. $f(x)=3x^2+2$. $\lim_{h\to 0} \frac{3(x+h)^2+2-3x^2-2}{h} \to \frac{\cancel{3x^2}+6xh+3h^2+\cancel{2}-\cancel{3x^2}-\cancel{2}}{h}$

$f'(x)=6x$

$\lim_{h\to 0} 6x+3h = 6x$

b. $f(x)=\dfrac{1}{1-x}$

$\lim_{h\to 0} \frac{\frac{1}{1-x-h} - \frac{1}{1-x}}{h} = \frac{\frac{\cancel{1}-\cancel{x}-\cancel{1}+\cancel{x}+h}{(1-x-h)(1-x)}}{h} = \frac{\cancel{h}}{(1-x-h)(1-x)} \cdot \frac{1}{\cancel{h}} = \lim_{h\to 0} \frac{1}{(1-x-h)(1-x)}$

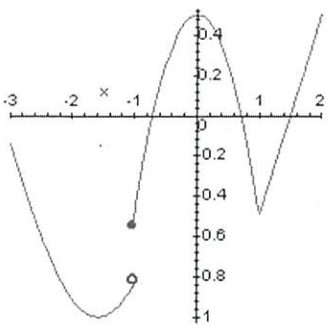
$f'(x) = \frac{1}{2(1-x)}$

c. $f(x)=\sqrt{x}$

$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{x}}$

**86.** Consider the function whose graph you see below, and find a number $x = c$ such that




a) $f$ is not continuous at $x = a$ $\quad x = -1$

b) $f$ is continuous but not differentiable at $x = b$, $\quad x = 1$

c) $f'$ is positive at $x = c$, $\quad x = 2$ (anywhere the slope is positive)

d) $f'$ is negative at $x = d$ $\quad x = -3$ (decreasing slope)

e) $f'$ is zero at $x = e$ $\quad x = 0$ (max or min)

f) $f'$ does not exist at $x = f$

$x = -1,\ x = 1$

**87.** Please find the derivative for each of the following functions (do not simplify unless you think it is helpful).

a. $f(x) = \pi^2 + x^2 + \sin(x) + \sqrt{x}$

skip

b. $f(x) = x^2(x^4 - 2x)$

$f'(x) = 6x^5 - 6x^2$

work on next page

c. $f(x) = x^2(x^3 - \frac{1}{x})$

$f(x) = x^5 - x$

$f'(x) = 5x^4 - 1$

d. $f(x) = 3x^5 - 2x^3 + 5x - \sqrt{2}$

$f'(x) = 15x^4 - 6x^2 + 5$

e. $f(x) = \dfrac{x^4 - 2x + 3}{x^2}$

$f'(x) = \dfrac{3x^4 + 2x - 6}{x^2}$

f. $f(x) = \sin^2(x)$

skip.

see next few pages for work!

87 a

$$\lim_{h \to 0} \frac{\pi^2 + (x+h)^2 + \sin(x+h) + \sqrt{x+h} - \pi^2 - x^2 - \sin x - \sqrt{x}}{h}$$

$$\cancel{\pi^2} + \cancel{x^2} + 2xh + h^2 + \sin(x+h) + \sqrt{x+h} - \cancel{\pi^2} - \cancel{x^2} - \sin x - \sqrt{x}$$

$$\frac{2xh + h^2 + \sin x \cosh + \cos x \sin h + \sqrt{x+h} - \sqrt{x} - \sin x}{h}$$

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87b

$$\lim_{h \to 0} \frac{(x+h)^2\left((x+h)^4 - 2(x+h)\right) - x^2(x^4 + 2x)}{h}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 1 | 1 | | | | | |
| 1 | 2 | 1 | | | | |
| 1 | 3 | 3 | 1 | | | |
| 1 | 4 | 6 | 4 | 1 | | |
| 1 | 5 | 10 | 10 | 5 | 1 | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 |

$$\lim_{h \to 0} \frac{(x+h)^6 - 2(x+h)^3 - x^6 + 2x^3}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^6} + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6 \cancel{- 2x^3} - 6x^2h - 6xh^2 - 2h^3 \cancel{- x^6} \cancel{+ 2x^3}}{\cancel{h}}$$

$$\lim_{h \to 0} \frac{6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5 - 6x^2 - 6xh - 2h^2}{h}$$

$$\boxed{f'(x) = 6x^5 - 6x^2}$$

87c

$$\lim_{h\to 0}\frac{(x+h)^5-(x+h)\overbrace{-x^5+x}^{-(x^5-x)}}{h}$$

1 5 10 10 5 1

$$\lim_{h\to 0}\frac{\cancel{x^5}+5x^4\cancel{h}+10x^3h^{\overset{1}{\cancel{2}}}+10x^2h^{\overset{2}{\cancel{3}}}+5xh^{\overset{3}{\cancel{4}}}+h^{\overset{4}{\cancel{5}}}-\cancel{x}-\overset{1}{\cancel{h}}-\cancel{x^5}+\cancel{x}}{\cancel{h}}$$

$$\lim_{h\to 0}\ 5x^4+10x^3h+10h^2x^2+5xh^3+h^4-1$$

$$f'(x)=5x^4-1$$

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87d

$$\lim_{h\to 0}\frac{3(x+h)^5-2(x+h)^3+5(x+h)\cancel{-\sqrt{2}}-3x^5+2x^3-5x\cancel{+\sqrt{2}}}{h}$$

$$\lim_{h\to 0}\frac{3(x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)-2(x^3+3x^2h+3xh^2+h^3)+5x+5h-3x^5+2x^3-5x}{h}$$

$$\lim_{h\to 0}\frac{\cancel{3x^5}+15x^4\cancel{h}+30x^3h^{\overset{1}{\cancel{2}}}+30x^2h^{\overset{2}{\cancel{3}}}+15xh^{\overset{3}{\cancel{4}}}+h^{\overset{4}{\cancel{5}}}-\cancel{2x^3}-6x^2\cancel{h}-6xh^{\overset{1}{\cancel{2}}}-2h^{\overset{2}{\cancel{3}}}+\cancel{5x}+5\cancel{h}-\cancel{3x^5}+\cancel{2x^3}-\cancel{5x}}{\cancel{h}}$$

$$\lim_{h\to 0}\ \underline{15x^4}+30x^5h+30x^2h^2+15xh^3+h^4\underline{-6x^2}-6xh-2h^2\underline{+5}$$

$$f'(x)=15x^4-6x^2+5$$

$\boxed{87e}$

$$\lim_{h\to 0} \frac{\dfrac{x\left((x+h)^4-2(x+h)+3\right)}{x\left((x+h)^2\right)} - \dfrac{\left(x^4-2x+3\right)\left(x^2+2xh+h^2\right)}{\left(x^2\right)(x+h)^2}}{h}$$

$$\lim_{h\to 0} \frac{\dfrac{\overset{\cancel{x^6}+4x^5h+6x^4h^2+4x^3h^3+x^2h^4-\cancel{2x^3}-2x^2h+\cancel{3x^2}}{x^2\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4-2x-2h+3\right)}-\cancel{x^6}+\cancel{2x^3}-\cancel{3x^2}-2x^5h+4x^2h-6xh-x^4h^2+2xh^2-3h^2}{(x)(x+h)^2}}{h}$$

$$\lim_{h\to 0} \frac{\dfrac{4x^5\cancel{h}+6x^4h^{\cancel{2}\,1}+4x^3h^{\cancel{3}\,2}+x^2h^{\cancel{4}\,3}-2x^2\cancel{h}-2x^5\cancel{h}+4x^2\cancel{h}-6x\cancel{h}-x^4h^{\cancel{2}\,1}+2xh^{\cancel{2}\,1}-3h^{\cancel{2}\,1}}{(x)(x+h)^2}}{\cancel{h}}$$

$$\lim_{h\to 0} \frac{4x^5+6x^4h+4x^3h^2+x^2h^3-2x^2-2x^5+4x^2-6x-x^4h+2xh-3h}{(x)(x+h)^2}$$

$$\frac{4x^5-2x^2-2x^5+4x^2-6x}{x^3}$$

$$f'(x)=\frac{3x^5+2x^2-6x}{x^3}$$

$$\boxed{f'(x)=\frac{3x^4+2x-6}{x^2}}$$

**88.** Find the equation of the tangent line to the function at the given point:

a) $f(x) = x^2 - x + 1$, at x = 0

$\boxed{\text{① Find derivative}}$

$f'(x) = \lim_{h \to 0} \dfrac{(x+h)^2 - (x+h) + 1 - (x^2 - x + 1)}{h}$

$\dfrac{x^2 + 2xh + h^2 - x - h + 1 - x^2 - x - 1}{h}$

$\boxed{\text{② Use point slope formula}}$

$f'(x) = 2x - 1$ $\quad f(0) = 1$ $\quad$ point $(0, 1)$

$f'(0) = -1$ $\quad$ slope $= -1$

$y - 1 = -1(x - 0)$

$\boxed{y = -x + 1}$

b) $f(x) = x^3 - 2x$, at x = 1

$\boxed{\text{① Find derivative}}$

$f'(x) = \lim_{h \to 0} \dfrac{(x+h)^3 - 2(x+h) - (x^3 - 2x)}{h}$

$= \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h - x^3 + 2x}{h}$

$f'(x) = 3x^2 - 2$

$\boxed{\text{② Point-Slope Formula at } x = 1}$

$f'(1) = 1 \leftarrow$ slope $\quad$ point $\rightarrow f(1) = -1$ $\quad (1, -1)$

$y + 1 = 1(x - 1)$

$\boxed{y = x - 2}$

**89.** Suppose the function $f(x) = \dfrac{x^4 - 2x + 3}{x^2}$ indicates the position of a particle.

① Simplify: $x^2 - \frac{2}{x} + \frac{3}{x^2}$

a) Find the velocity after 10 seconds First derivative

$f'(x) = 2x + \frac{2}{x^2} - \frac{6}{x^3}$ $\quad f'(10) = 20 + \frac{2}{100} - \frac{6}{1000} = 20.014$

b) Find the acceleration after 10 seconds second derivative (Derivative of the first derivative)

$f''(x) = 2 - \frac{4}{x^3} + \frac{18}{x^4}$ $\quad f''(10) = 2 - \frac{4}{1000} + \frac{18}{10000} = 1.9978$

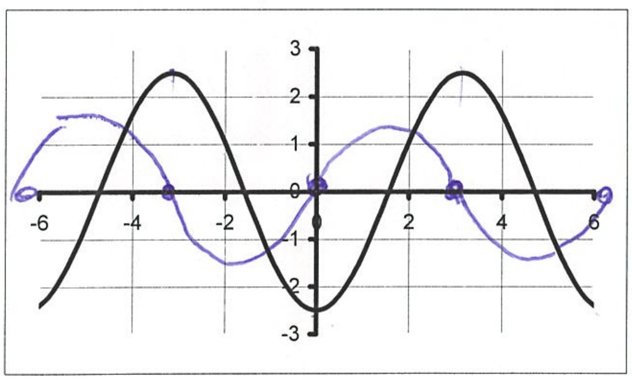
c) When is the particle at rest (other than for t = 0)

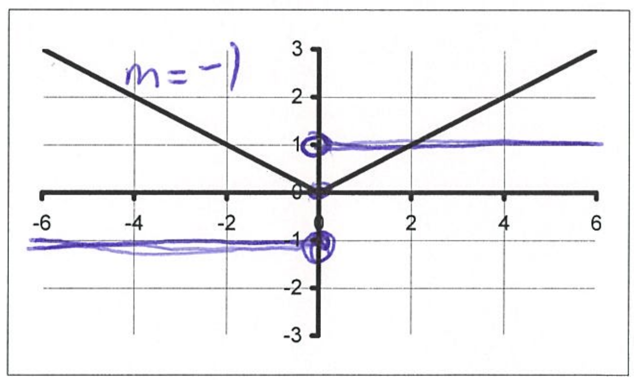
$f'(x) = 2x + \frac{2}{x^2} - \frac{6}{x^3}$ $\quad$ (when derivative is zero)

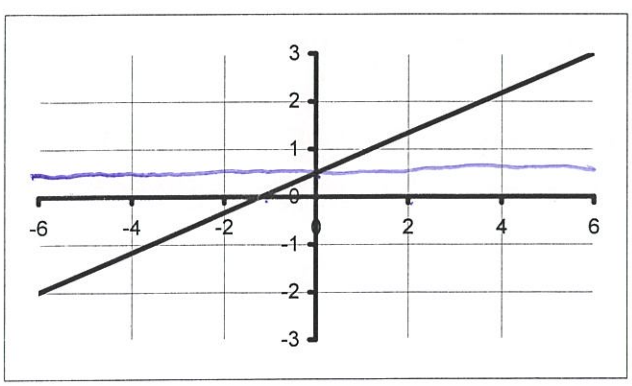
d) When is the particle moving forward and when backward

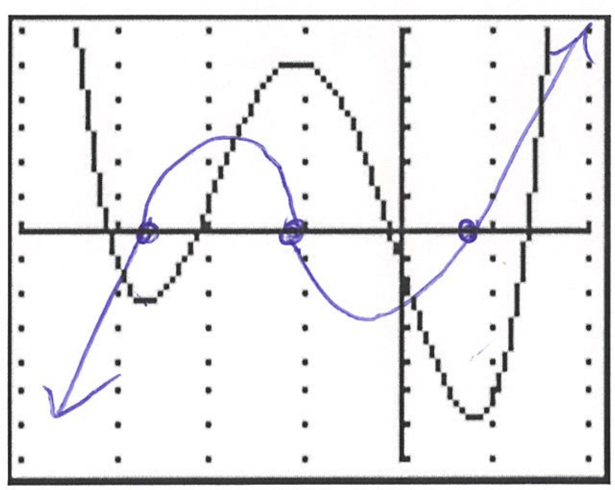
(when derivative is neg/pos)

**90. Sketch the graph of the derivative of each of the following functions on the same graph.**





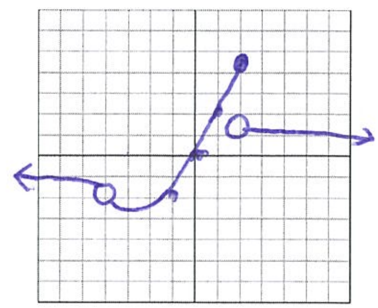





**91. Draw a graph with the following conditions.** answers may vary.

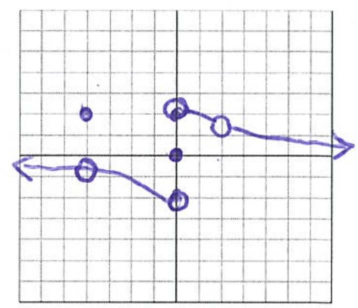
**Function #1**

- f(0) = 0
- f(1) = 2
- f(-1) = -2
- at f(3) there is a non-removable discontinuity
- at f(-4) there is a removable discontinuity
- $\lim_{x\to-\infty} f(x) = -1$
- $\lim_{x\to\infty} f(x) = 1$



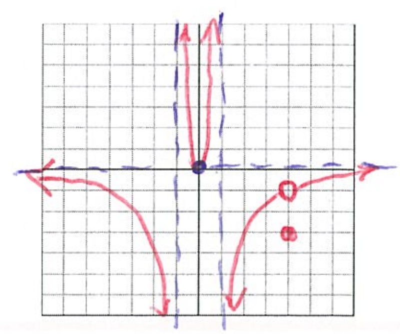
**Function #2**

- $\lim_{x\to\infty} f(x) = 0$
- $\lim_{x\to-\infty} f(x) = 0$
- f(0) = 0
- at f(-4) there is a removable discontinuity
- $\lim_{x\to 2} f(x)$ exists, but the graph is discontinuous
- $\lim_{x\to 0^+} f(x) = 2$
- $\lim_{x\to 0^-} f(x) = -2$



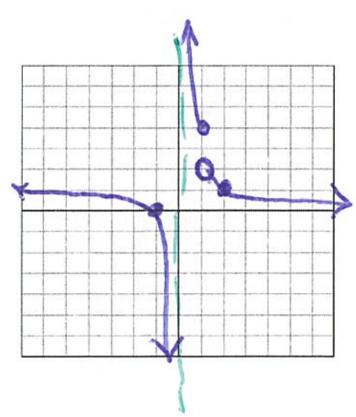
**Function #3**

- $\lim_{x\to\infty} f(x) = 0$
- $\lim_{x\to-\infty} f(x) = 0$
- f(0) = 0
- $\lim_{x\to 4} f(x)$ exists, but the graph is discontinuous
- $\lim_{x\to 1^+} f(x) = -\infty$
- $\lim_{x\to -1^-} f(x) = -\infty$
- $\lim_{x\to 1^-} f(x) = \lim_{x\to -1^+} f(x) = \infty$



**Function #4**

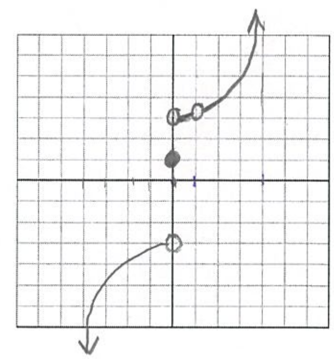
- $\lim_{x\to\infty} f(x) = 0$
- $\lim_{x\to-\infty} f(x) = 1$
- f(-1) = 0
- $\lim_{x\to 1} f(x)$ does not exist
- $\lim_{x\to 0^+} f(x) = \infty$
- $\lim_{x\to 0^-} f(x) = -\infty$
- f(2) = 1



**Function #5**

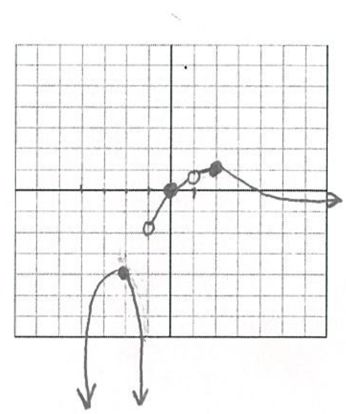
- f(-3) = 0
- $\lim\limits_{x\to\infty} f(x) = -1$
- $\lim\limits_{x\to-2^+} f(x) = \lim\limits_{x\to-2^-} f(x) = \infty$
- at f(5) there is a removable discontinuity
- $\lim\limits_{x\to1} f(x)$ does not exist
- f(0) = 3
- $\lim\limits_{x\to-4^+} f(x) = \infty$

**Function #6**



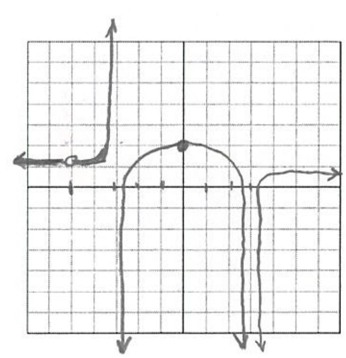
- $\lim\limits_{x\to0^+} f(x) = 3$
- $\lim\limits_{x\to0^-} f(x) = -3$
- f(0) = 1
- $\lim\limits_{x\to1} f(x)$ exists, but the graph is discontinuous
- $\lim\limits_{x\to-4^+} f(x) = -\infty$
- $\lim\limits_{x\to4^-} f(x) = \infty$

**Function #7**



- f(0) = 0
- f(2) = 1
- f(-2) = -4
- $\lim\limits_{x\to-1} f(x)$ does not exist
- $\lim\limits_{x\to1} f(x)$ exists, but the graph is discontinuous
- $\lim\limits_{x\to\infty} f(x) = -1$
- $\lim\limits_{x\to-4^+} f(x) = -\infty$

**Function #8**



- $\lim\limits_{x\to\infty} f(x) = 1$
- $\lim\limits_{x\to3^+} f(x) = -\infty$
- $\lim\limits_{x\to-3^+} f(x) = -\infty$
- f(0) = 2
- at f(-5) there is a non-removable discontinuity
- $\lim\limits_{x\to-\infty} f(x) = 1$
- $\lim\limits_{x\to3^-} f(x) = -\infty$
- $\lim\limits_{x\to-3^-} f(x) = \infty$