4.6H Warm-up

Degrees to Radians

During the spring runoff of melting snow, the stream of water powering this waterwheel causes it to make one complete revolution counterclockwise every 3 seconds. $\frac{360}{3} = 120^\circ$

Write an equation to represent the height of a particular paddle of the waterwheel above or below the water level at any time, *t,* after the paddle emerges from the water.

18. Write your equation so the height of the paddle will be graphed correctly on a calculator set in *degree* mode.

$y = \sin(120t)$

19. Revise your equation so the height of the paddle will be graphed correctly on a calculator set in *radian* mode.

$y = \sin\left(\frac{2\pi}{3} t\right)$

During the summer months, the stream of water powering this waterwheel becomes a "lazy river" causing the wheel to make one complete revolution counterclockwise every 12 seconds. $\frac{360}{12} = 30^\circ$

Write an equation to represent the height of a particular paddle of the waterwheel above or below the water level at any time, *t,* after the paddle emerges from the water.

20. Write your equation so the height of the paddle will be graphed correctly on a calculator set in *degree* mode.

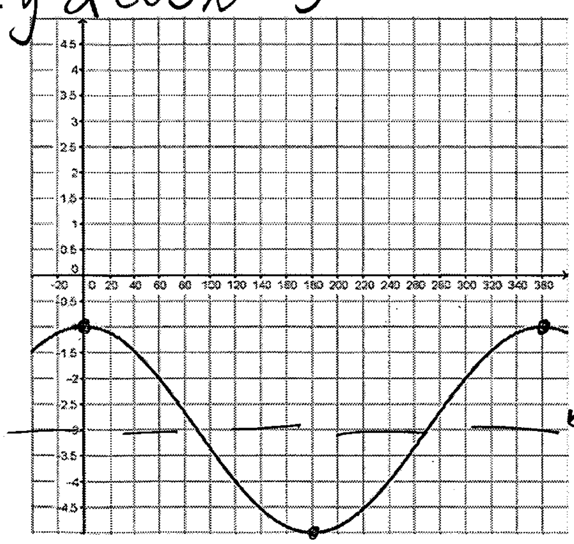
$y = \sin(30t)$

21. Revise your equation so the height of the paddle will be graphed correctly on a calculator set in *radian* mode.

$y = \sin\left(\frac{\pi}{6} t\right)$

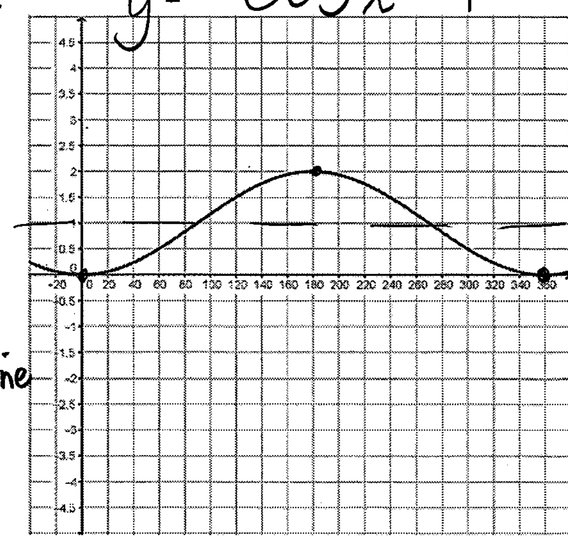
Given the graphs below, write at least one function that can be used to model the graph.

8. $y = 2\cos x - 3$



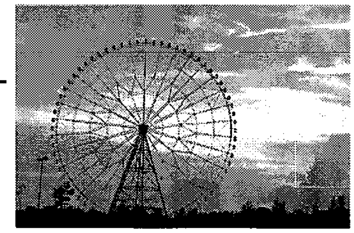
midline

9. $y = -\cos x + 1$



midline

4.6H High Noon and Sunset Shadows
A Develop Understanding Task



In this task, we revisit the Ferris wheel that caused Carlos so much anxiety. Recall the following facts from previous tasks:

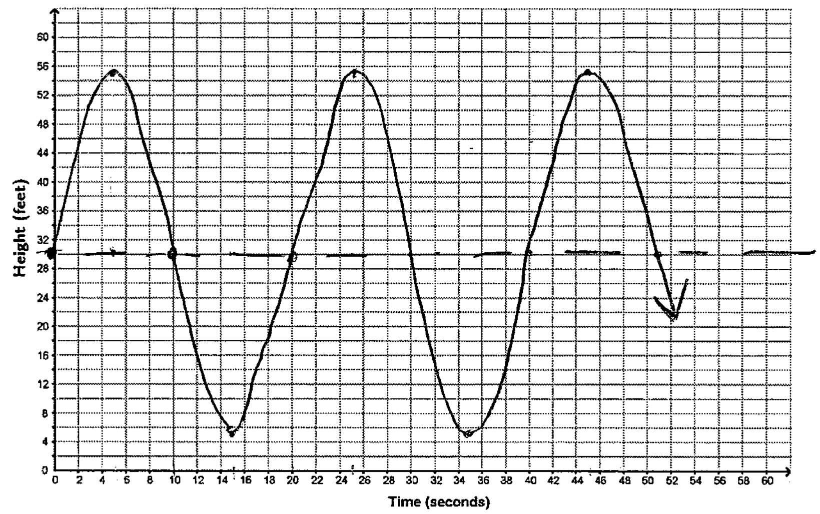
- The Ferris wheel has a radius of 25 feet.
- The center of the Ferris wheel is 30 feet above the ground.
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds.

$\frac{360}{20} = 18$

The Ferris wheel is located next to a high-rise office complex. At sunset, a rider casts a shadow on the exterior wall of the high-rise building. As the Ferris wheel turns, you can watch the shadow of the rider rise and fall along the surface of the building. In fact, you know an equation that would describe the height of this "sunset shadow."

1. Write the equation of this "sunset shadow."

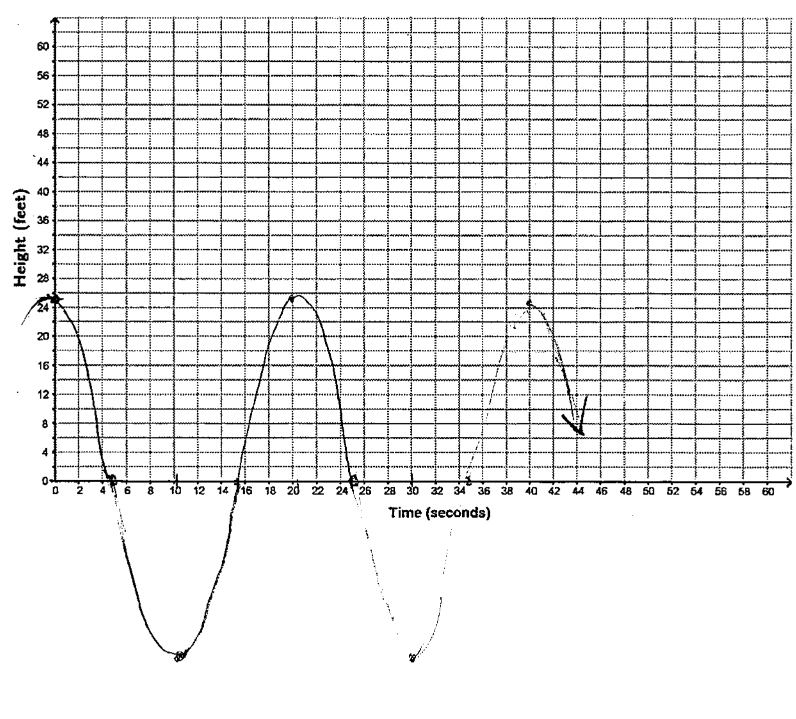
$y = 25\sin 18x + 30$



At noon, when the sun is directly overhead, a rider casts a shadow that moves left and right along the ground as the Ferris wheel turns. In fact, you know an equation that would describe the motion of this "high noon shadow."

2. Write the equation of this "high noon shadow."

$y = 25\cos 18x$



3. Based on your previous work, you probably wrote these equations in terms of the angle of rotation being measured in degrees. Revise your equations so the angle of rotation is measured in **radians.**

   a. The "sunset shadow" equation in terms of radians:

   $$y = 25\sin\frac{\pi}{10}x + 30$$

   b. The "high noon shadow" equation in terms of radians:

   $$y = 25\cos\frac{\pi}{10}x$$

4. In the equations you wrote in question 3, where on the Ferris wheel was the rider located at time $t = 0$? We will refer to the position as the rider's ***initial position*** on the wheel.

   a. Initial position for the "sunset" shadow equation:

   $(0, 30)$

   b. Initial position for the "high noon" shadow equation:

   $(25, 0)$

5. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **at the top of the wheel**. Sketch a graph of the new functions.

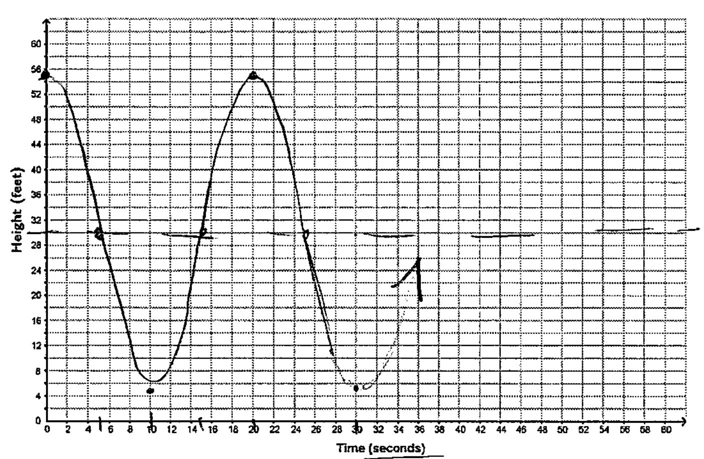
   a. The "sunset shadow" equation, initial position at the top of the wheel:

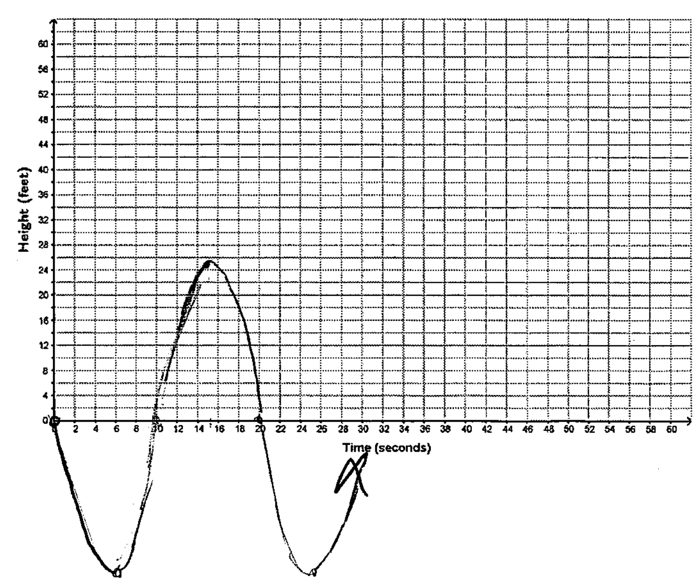
   $y = 25\sin\left(\frac{\pi}{10}x + \frac{\pi}{2}\right) + 30$ / $y = 25\sin\frac{\pi}{10}(x+5) + 30$

   * makes this the same as the cosine graph.

   b. The "high noon shadow" equation, initial position at the top of the wheel:

   $y = 25\cos\left(\frac{\pi}{10}x + \frac{\pi}{2}\right)$ / $y = 25\cos\frac{\pi}{10}(x+5)$





$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

(*) Look on desmos to check your graphs

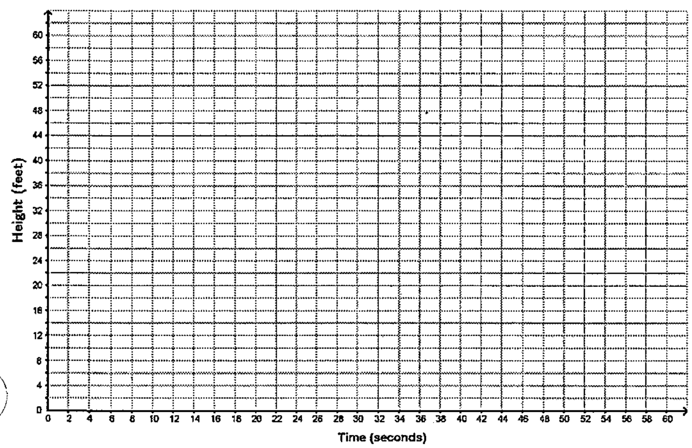
6. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **at the bottom of the wheel.**

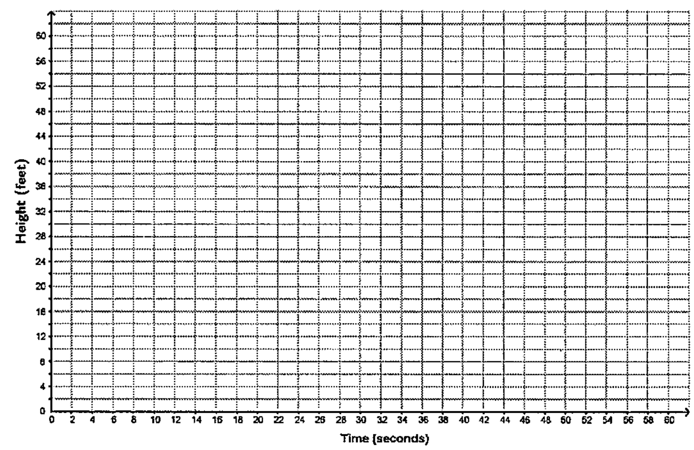
   a. The "sunset shadow" equation, initial position at the bottom of the wheel:

   $y = 25 \sin \frac{\pi}{10}(x+15)+30$ or $y=25\sin\frac{\pi}{10}(x-5)+30$

   b. The "high noon shadow" equation, initial position at the bottom of the wheel:

   $y = 25\cos\frac{\pi}{10}(x-5)$ * shift right 5 units





7. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **at the point farthest to the left of the wheel.**

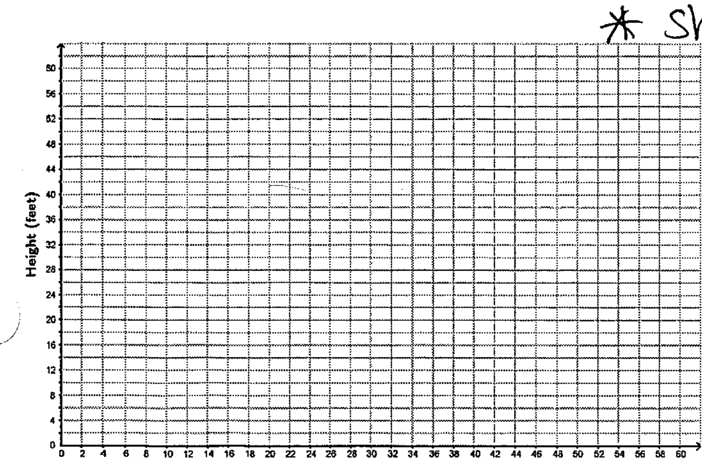
   a. The "sunset shadow" equation, initial position at the point farthest to the left of the wheel:

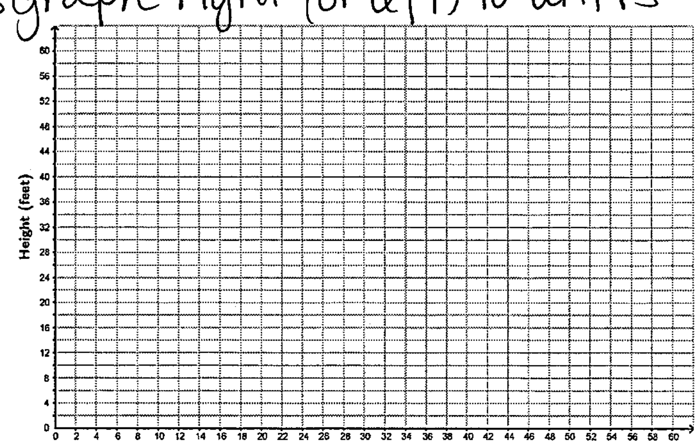
   $y=25\sin\frac{\pi}{10}(x+10)$ or $y=25\sin\frac{\pi}{10}(x-10)$

   b. The "high noon shadow" equation, initial position at the point farthest to the left of the wheel:

   $y=25\cos\frac{\pi}{10}(x+10)$ or $y=25\cos\frac{\pi}{10}(x+10)$

   * shifts graph right (or left) 10 units





8. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **halfway between the farthest point to the right on the wheel and the top of the wheel.**

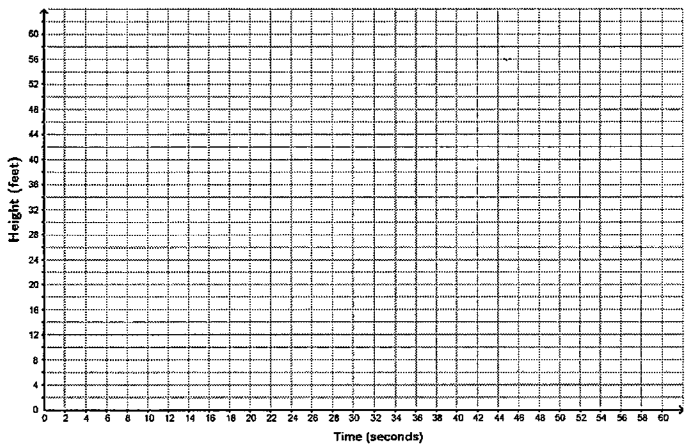
   a. The "sunset shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

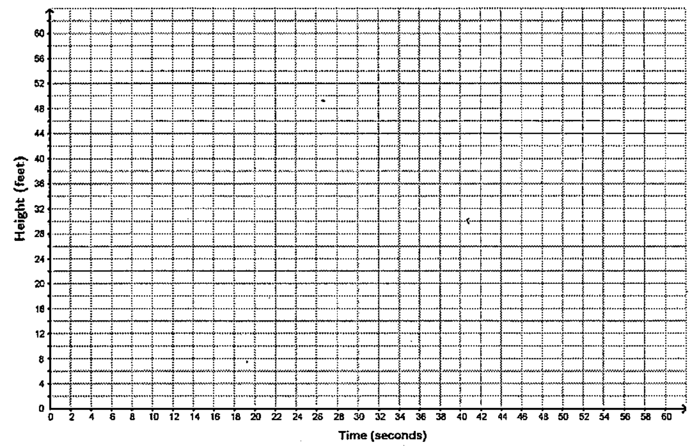
   $$y = 25\sin\frac{\pi}{10}(x+2.5)+30$$

   b. The "high noon shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

   $$y = 25\cos\frac{\pi}{10}(x+2.5)$$

   * shifts left 2.5 units





9. Revise your equations from question 3 so that the wheel **rotates twice as fast.**

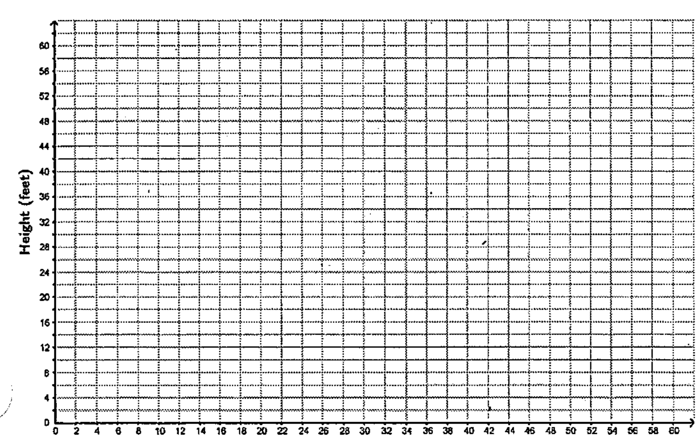
   a. The "sunset shadow" equation for the wheel rotating twice as fast:

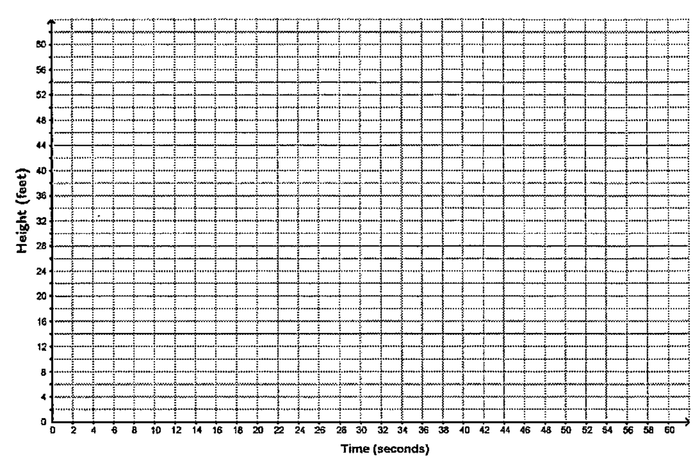
   $$y = 25\sin\frac{\pi}{5}x+30$$

   b. The "high noon shadow" equation for the wheel rotating twice as fast:

   *period is 10 instead of 20

   $$y = 25\cos\frac{\pi}{5}x$$





radius = 25 ft
center = 60 ft above ground

10. Revise your equations from question 3 so that the **radius of the wheel is twice as large and the center of the wheel is twice as high.**

a. The "sunset shadow" equation for a radius twice as large and the center twice as high:

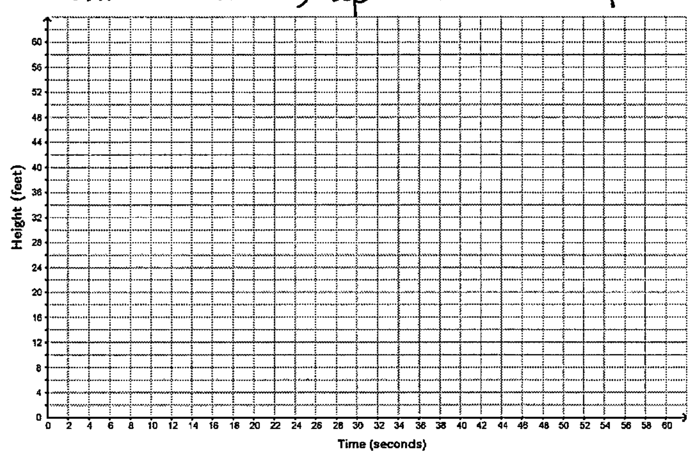
$$y = 50\sin\frac{\pi}{10}x + 60$$

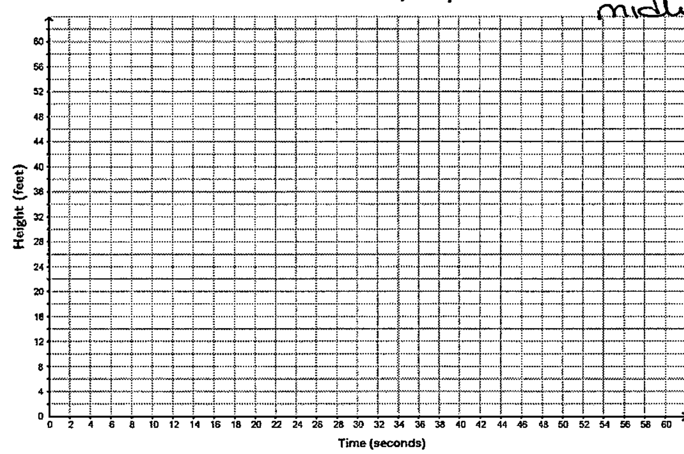
b. The "high noon shadow" equation for a radius twice as large and the center twice as high:

$$y = 50\cos\frac{\pi}{10}x$$

midline at 60, up and down 50 from midline

* midline at 0, up & down 50 from midline





11. Below is the standard form for a sine and cosine trigonometric function. Explain how to find the following features on a graph using standard form.

$y = A\sin(Bx - C) + D$ | $y = A\cos(Bx - C) + D$

a. y-intercept: depends on phase shift

$(0, D)$ | $(0, D + A)$

b. Midline(Vertical Shift):

$y = D$ | $y = D$

c. Amplitude:

$|A|$ | $|A|$

d. Period:

$\frac{2\pi}{B}$ or $\frac{360}{B}$ | $\frac{2\pi}{B}$ or $\frac{360}{B}$

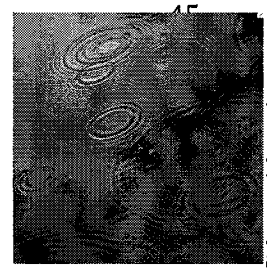
e. Phase (Horizontal) Shift:
* Factor the B value out

$\frac{C}{B}$ phase shift

example: $3\sin(2x - 6) + 5$
$3\sin(2(x - 3)) + 5$ * right 3 units

4.7H Warm Up

Getting on the Right Wavelength - A Practice Understanding Task



Below is a new Ferris wheel that has a $\boxed{\text{radius of 40 feet}}$ whose $\boxed{\text{center is 50 feet from the}}$ ground, and makes one revolution counterclockwise every 18 seconds.

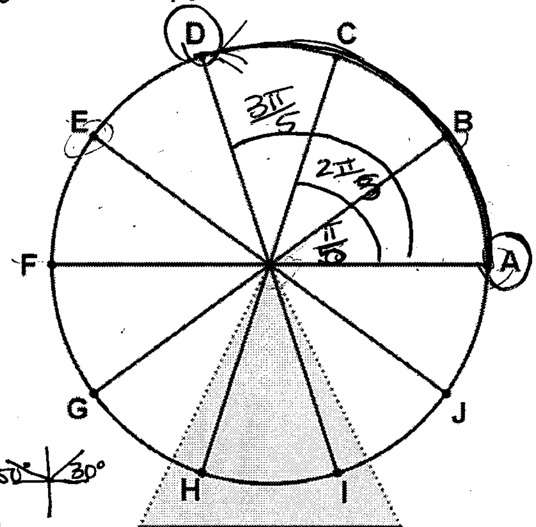
$\frac{360}{18} = 20 \qquad \frac{2\pi}{18} = \frac{\pi}{9}$

1. Write the equation of the height from the ground of the rider at any time t, if at $t = 0$ the rider is at **position A.** Use radians to measure the angles of rotation.

$y = 40\sin 20x + 50$

or

$y = 40\sin \frac{\pi}{9}x + 50$



2. At what time(s) is the rider 70 feet above the ground? Show the details of how you answered this question.

$70 = 40\sin 20x + 50$

$20 = 40\sin 20x$

$\frac{1}{2} = \sin 20x \qquad u = 20x$

$\frac{1}{2} = \sin u$

$u = 30 \quad u = 150$

$20x = 30 \quad 20x = 150$

$\boxed{x = 1.5\text{s} \quad x = 7.5 \text{ seconds}}$

3. Use your answer from question 2 to write an equation to show when the rider is 70 feet above the ground if the Ferris wheel goes around forever.

$x = 1.5 + 18n$

$x = 7.5 + 18n$

4. If you used a sine function in question 1, revise your equation to model the same motion with a cosine function. If you used a cosine function in question 1, revise your equation to model the motion with a sine function.

$y = 40\cos(20x - 90) + 50$

let x = t

5. Write the equation of the height of the rider at any time t, if at $t = 0$, the rider is at **position** D. Use radians to measure the angles of rotation.

$$y = 40\sin\left(\frac{\pi}{9}x + \frac{3\pi}{5}\right) + 50$$

or

$$y = 40\sin\left(\frac{\pi}{9}\left(x + \frac{27}{5}\right)\right) + 50$$

6. For the equation you wrote in ~~question 4,~~ question 5 at what time(s) is the rider 80 feet above the ground? Show or explain the details of how you answered this question.

$$80 = 40\sin\left(\frac{\pi}{9}x + \frac{3\pi}{5}\right) + 50 \qquad u = \frac{\pi}{9}x + \frac{3\pi}{5}$$

$$30 = 40\sin\left(\frac{\pi}{9}x + \frac{3\pi}{5}\right) \qquad 2.29353 = \frac{\pi}{9}x + \frac{3\pi}{5} \qquad .848062 = \frac{\pi}{9}x + \frac{3\pi}{5}$$

$\pi - \alpha$ $\qquad \alpha$ .848062

$$\frac{3}{4} = \sin\left(\frac{\pi}{9}x + \frac{3\pi}{5}\right) \qquad x = 1.17 \text{ s} \qquad x = -2.97 \text{ s}$$

2.29353

$$\frac{3}{4} = \sin u \qquad u = .848062$$

7. Use your answer from question 6 to write an equation to show when the rider is 80 feet above the ground if the Ferris wheel goes around forever.

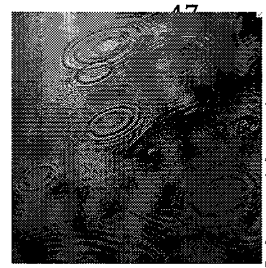
$$\frac{2\pi}{\frac{\pi}{9}} = 2\pi \cdot \frac{9}{\pi}$$

$$x = 1.17 + 18n$$

$$x = -2.97 + 18n$$

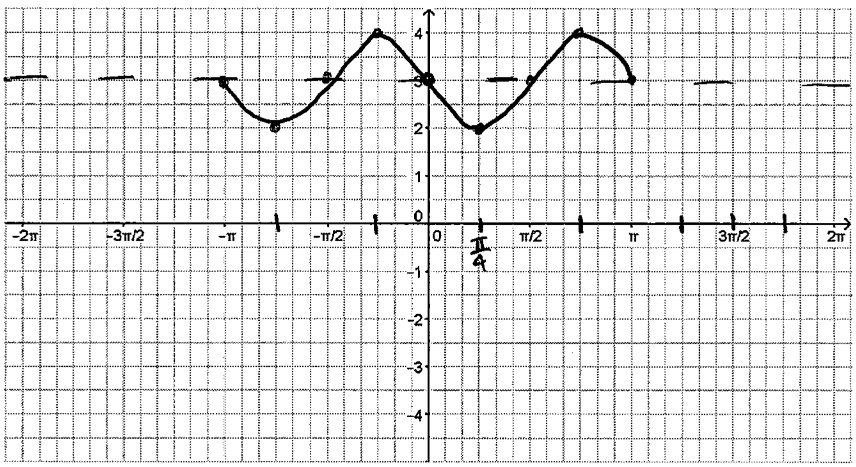
4.7H *Graphing Sine and Cosine*

A Practice Understanding Task



Graph the following functions $\boxed{\text{using radians}}$ and identify the listed features.

1. $y = -\sin 2x + 3$



Domain: $(-\infty, \infty)$

Range: $[2, 4]$

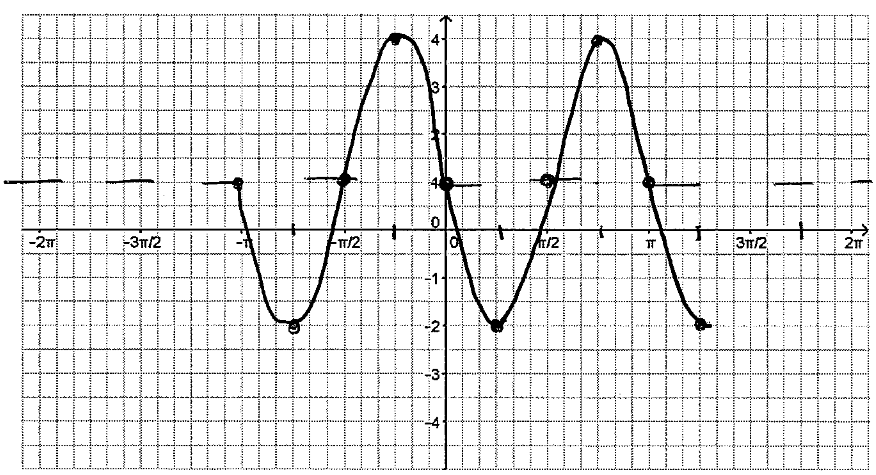
Amplitude: 1

Period: $\frac{2\pi}{2} = \pi$ scale: $\frac{\pi}{4}$

Phase Shift: none

Midline: 3

2. $y = -3\sin 2x + 1$



Domain: $(-\infty, \infty)$

Range: $[-2, 4]$

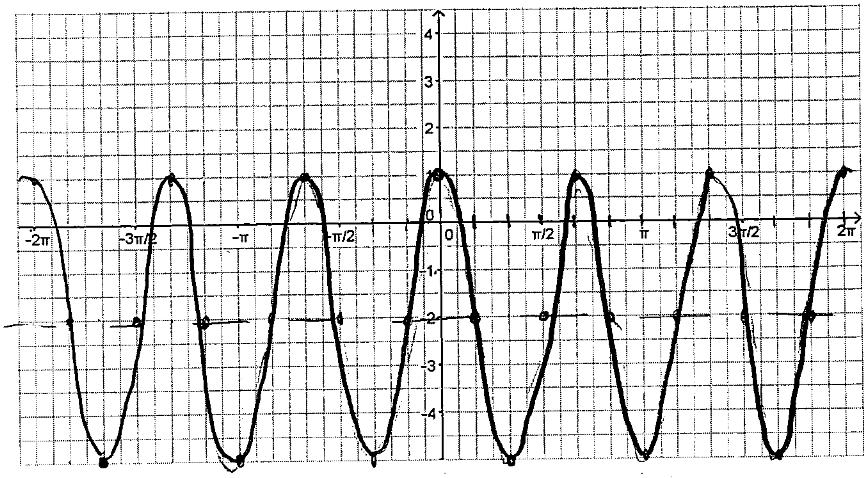
Amplitude: 3

Period: $\frac{2\pi}{2} = \pi$ scale: $\frac{\pi}{4}$

Phase Shift: none

Midline: 1

3. $y = 3\cos 3x - 2$



Domain: $(-\infty, \infty)$

Range: $[-5, 1]$

Amplitude: 3

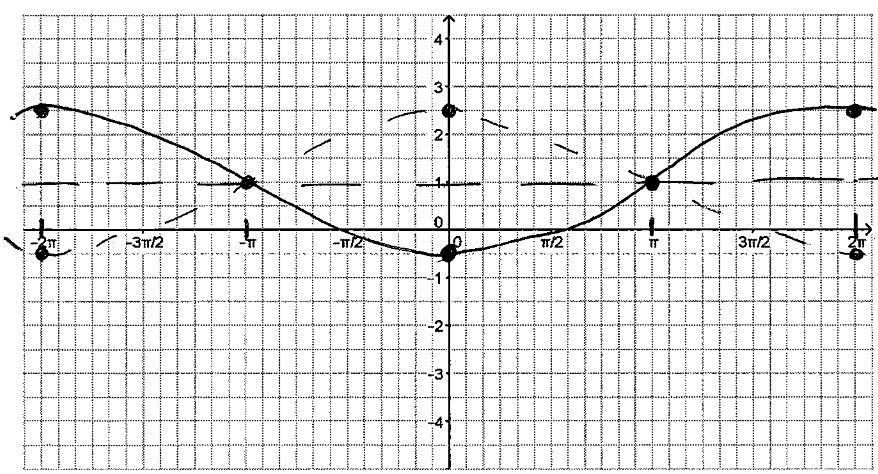
Period: $\frac{2\pi}{3}$ scale: $\frac{\pi}{6}$

Phase Shift: none

Midline: -2

$y = \frac{3}{2}\cos\frac{1}{2}(x - 2\pi) + 1$

4. $y = \frac{3}{2}\cos\left(\frac{x}{2} - \pi\right) + 1$



Domain: $(-\infty, \infty)$

Range: $[-.5, 1.5]$

Amplitude: $\frac{3}{2}$

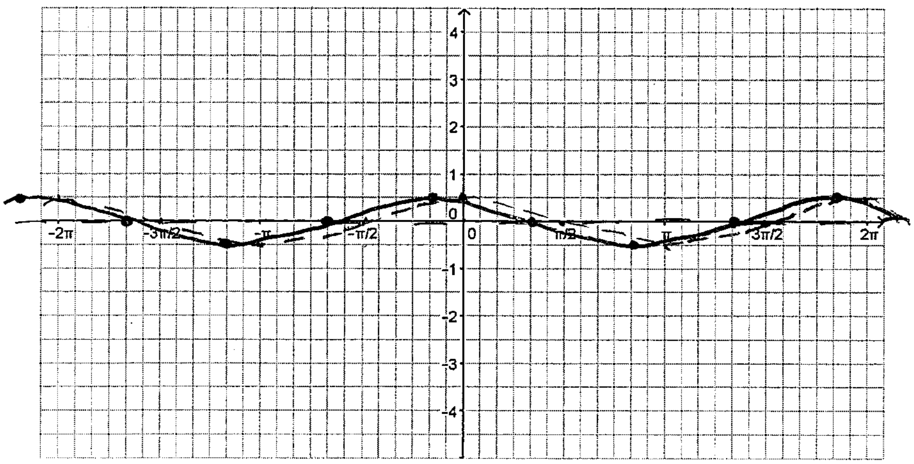
Period: $2\pi \cdot 2 = 4\pi$ scale $\frac{4\pi}{4} = \pi$

Phase Shift: right 2π

Midline: 1

* It may be easier to graph the curve without phase shift first with a dotted line, then include the phase shift as shown above.

5. $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{6}\right)$



Domain: $(-\infty, \infty)$

Range: $[-.5, .5]$

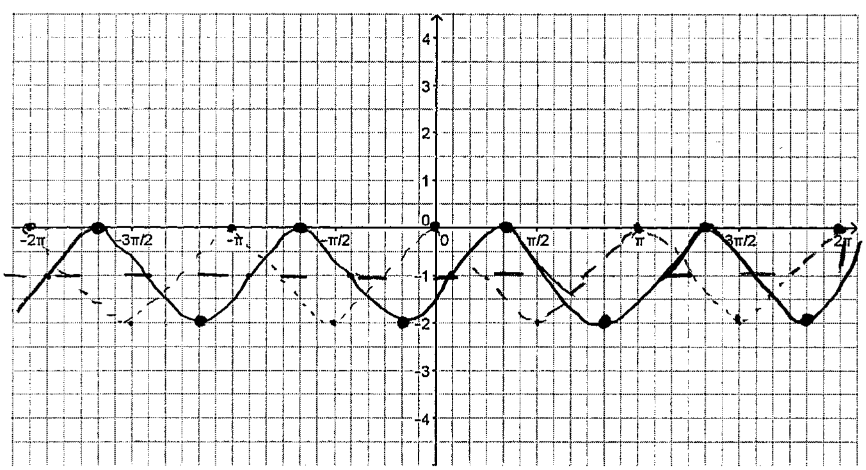
Amplitude: $1/2$

Period: 2π

Phase Shift: left $\pi/6$

Midline: 0

6. $y = \cos\left(2\left(x - \frac{\pi}{3}\right)\right) - 1$



Domain: $(-\infty, \infty)$

Range: $[-2, 0]$

Amplitude: 1

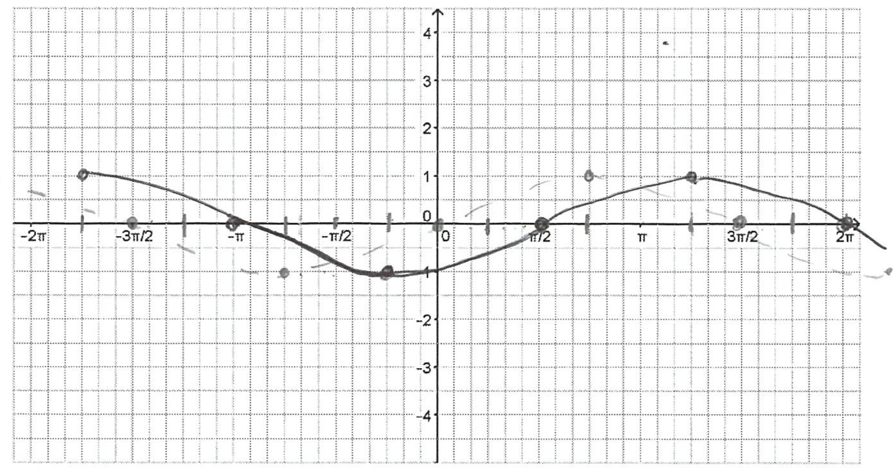
Period: π

Phase Shift: right $\pi/3$

Midline: -1

$\frac{\cancel{2}\pi}{1} \cdot \frac{1}{\cancel{2}} = \pi$

7. $y = \sin\left(\frac{2}{3}\left(\theta - \frac{\pi}{2}\right)\right)$



Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Amplitude: 1

Period: $\frac{2\pi}{\frac{2}{3}} = 2\pi \cdot \frac{3}{2} = 3\pi$

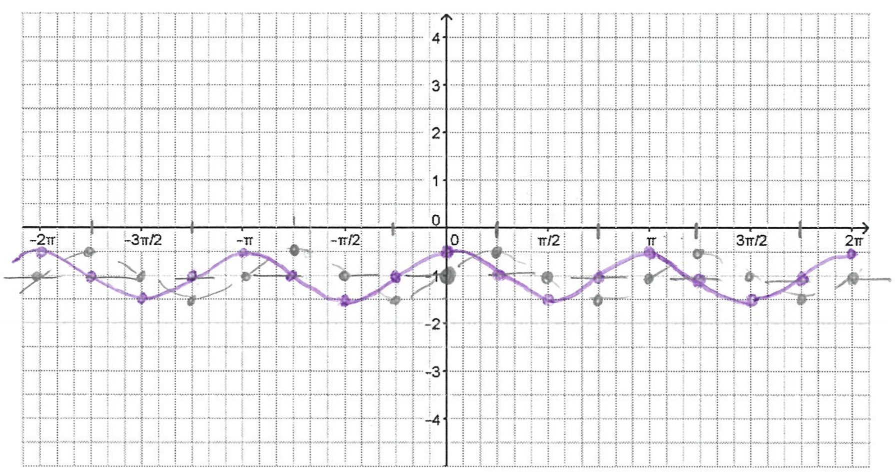
scale: $\frac{3\pi}{4}$

Phase Shift: right $\frac{\pi}{2}$

Midline: 0

$\frac{1}{2}\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$

8. $y = \frac{1}{2}\sin\left(2x + \frac{\pi}{2}\right) - 1$



Domain: $(-\infty, \infty)$

Range: $[-1.5, -.5]$

Amplitude: $\frac{1}{2}$

Period: $\frac{2\pi}{2} = \pi$

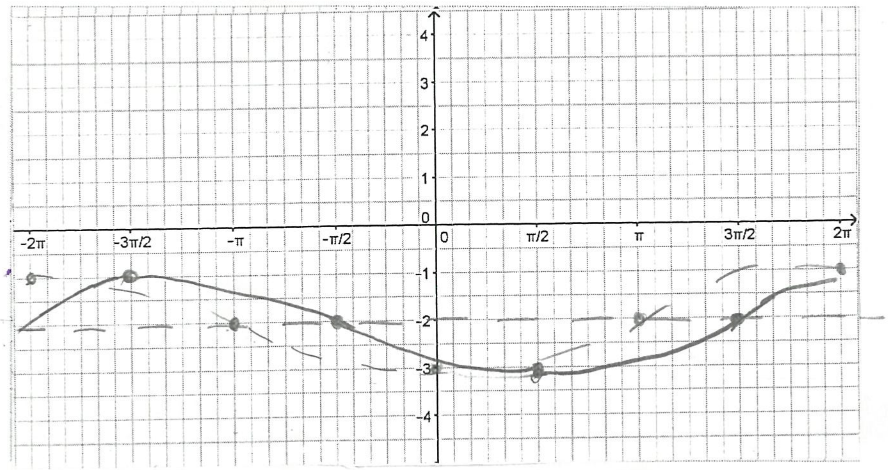
sale: $\frac{\pi}{4}$

Phase Shift: left $\frac{\pi}{4}$

Midline: -1

$-\cos\frac{1}{2}x-2$

9. $y=-\cos\left(\frac{1}{2}\left(x-\frac{\pi}{2}\right)\right)-2$



Domain: $(-\infty, \infty)$

Range: $[-3, -1]$

Amplitude: 1

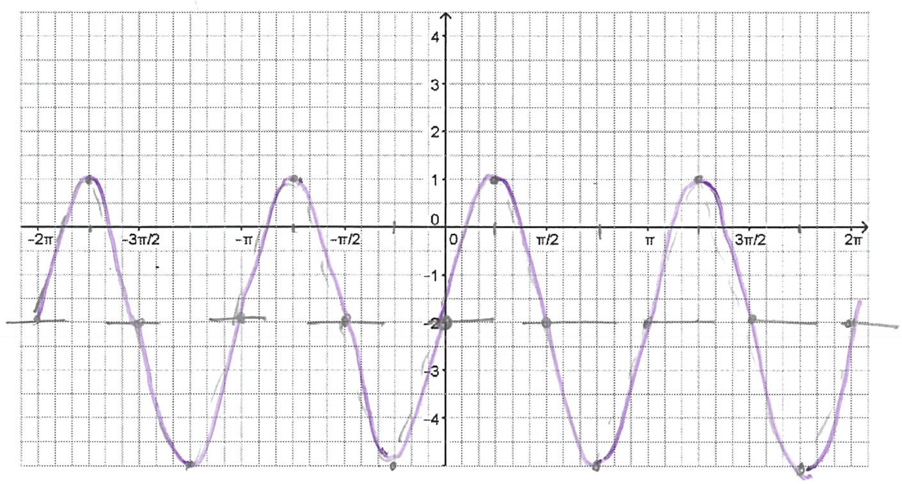
Period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$ scale: π

Phase Shift: right $\frac{\pi}{2}$

Midline: -2

$y=3\sin(2(x-\pi))-2$

10. $y=3\sin(2x-2\pi)-2$



Domain: $(-\infty, \infty)$

Range: $[-5, 1]$

Amplitude: 3

Period: $\frac{2\pi}{2} = \pi$ scale: $\frac{\pi}{4}$

Phase Shift: right π

Midline: -2

4.8H Warm Up

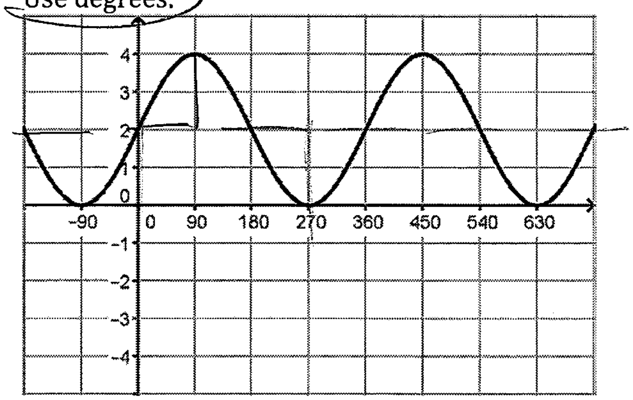
Transformations of Sine and Cosine & Solving trigonometric equations

Write a sine and a cosine equation for each of the graphs below.

$\frac{360}{B} = 270$

$\frac{4}{3} = \frac{360}{270} = B$

1. Use degrees.



midline = 2

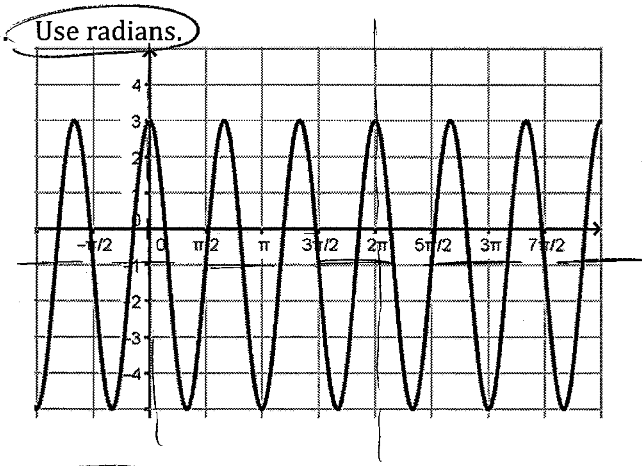
amp = 2

period = 270

$90 \cdot \frac{3}{4}$

$$\boxed{\begin{array}{l} y = 2\sin\left(\frac{4}{3}x\right) + 2 \\ y = 2\cos\left(\frac{4}{3}x - 90\right) + 2 \\ \text{or} \\ y = 2\cos\frac{4}{3}\left(x - \frac{135}{2}\right) + 2 \end{array}}$$

2. Use radians.



period = $\frac{2\pi}{3}$

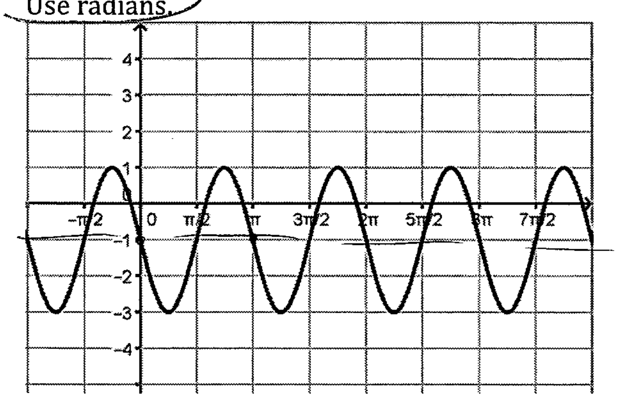
amp = 4

midline = -1

$$\boxed{\begin{array}{l} y = 4\cos(3x) - 1 \\ y = 4\sin\left(3x + \frac{\pi}{2}\right) - 1 \\ \text{or} \\ y = 4\sin\left(3\left(x + \frac{3\pi}{2}\right)\right) - 1 \end{array}}$$

$\frac{2\pi}{\pi} = 2$

3. Use radians.



period = π

amp = 2

mid = -1

$$\boxed{\begin{array}{l} y = -2\sin 2x - 1 \\ y = -2\cos\left(2x - \frac{\pi}{2}\right) - 1 \\ \text{or} \\ y = -2\cos\left(2\left(x - \frac{\pi}{4}\right)\right) - 1 \end{array}}$$

Solve each of the equations below and write the general solution.

4. $2\sin 3x \sec 3x = \sec 3x$

$2\sin 3x \sec 3x - \sec 3x = 0$

$\sec(3x)(2\sin 3x - 1) = 0$

$\sec(3x) = 0$ $\quad\quad$ $\sin 3x = \frac{1}{2}$

$\cancel{\frac{1}{\cos x} = 0}$ $\quad$ $u = 3x$

Not possible $\quad$ $\sin u = \frac{1}{2}$ $\quad$ $\frac{5\pi}{6}$ $\quad$ $\frac{\pi}{6}$

$3x = \frac{\pi}{6} + 2\pi n$ $\quad\quad$ $3x = \frac{5\pi}{6} + 2\pi n$

$\boxed{x = \frac{\pi}{18} + \frac{2\pi}{3}n}$ $\quad\quad$ $\boxed{x = \frac{5\pi}{18} + \frac{2\pi n}{3}}$

5. $\sin 2x + 2\cos x = 0$

Formula sheet →

$2\sin x\cos x + 2\cos x = 0$

$2\cos x(\sin x + 1) = 0$

$2\cos x = 0$ $\quad$ $\sin x = -1$

$\cos x = 0$ $\quad\quad$ $x = \frac{3\pi}{2}$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2} + 2\pi n}$

6. $tan^2x + 3\sec x + 3 = 0$

$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2}$

$\tan^2 + 1 = \sec^2$

$\tan^2 = \sec^2 - 1$

$(\sec^2 - 1) + 3\sec x + 3 = 0$

$\sec^2 + 3\sec x + 2 = 0$

$(\sec x + 2)(\sec x + 1) = 0$

$\sec x = -2$ $\quad$ $\sec x = -1$

$\cos x = -\frac{1}{2}$ $\quad$ $\cos x = -1$

$\frac{\pi}{3}$

$\boxed{\begin{array}{l} x = \frac{2\pi}{3} + 2\pi n \\ x = \frac{4\pi}{3} + 2\pi n \end{array}}$ $\quad$ $\boxed{x = \pi + 2\pi n}$

7. $4sin^4x + 3sin^2x - 1 = 0$

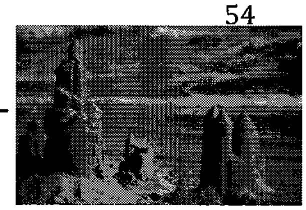
$(4\sin^2 - 1)(\sin^2 + 1) = 0$

$\sin^2 = \frac{1}{4}$ $\quad\quad$ $\cancel{\sin^2 = -1}$

$\sin x = \pm\frac{1}{2}$ $\quad\quad$ not possible

$\frac{\pi}{6}$ $\quad$ $\frac{\pi}{6}$

$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} + 2\pi n}$



4.8H High Tide

A Solidify Understanding Task

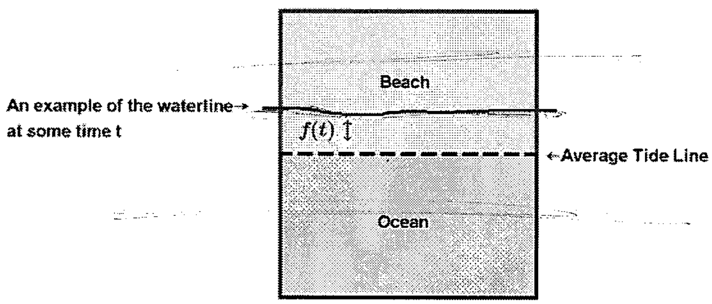
Perhaps you have built an elaborate sand castle at the beach only to have it get swept away by the in-coming tide.

Frustrated by having your sandcastle knocked down by the tide, you decide to pay attention to the tides so that you can keep track of how much time you have to build and admire your sand castle.

You have a friend who is in calculus who will be going to the beach with you. You give your friend some data from the almanac about high tides along the ocean, as well as a contour map of the beach you intend to visit, and ask her to come up with an equation for the water level on the beach on the day of your trip. According to your friend's analysis, the water level on the beach will fit this equation:

$$f(t) = 20\sin\left(\frac{\pi}{6}t\right)$$

In this equation, $f(t)$ represents how far the tide is "in" or "out" from the average tide line. This distance is measured in feet and t represents the elapsed time, in hours, since midnight.



1. What is the highest up the beach (compared to its average position) that the tide will be during the day? This is called ***high tide***. What is the lowest that the tide will be during the day? This is called ***low tide***.

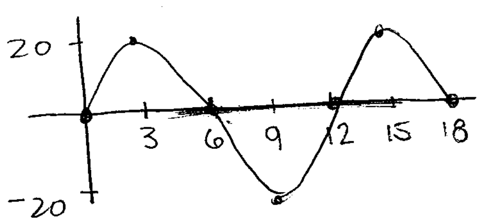
$f(t) = 20\sin\left(\frac{\pi}{6}t\right)$ Highest is 20 above average

Lowest is -20 below average

2. Suppose you plan to build your castle right on the average tide line just as the water has moved below that line. How much time will you have to build your castle before the incoming tide destroys your work?

$2\pi \cdot \frac{6}{\pi} = 12$

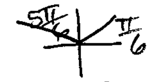
$12/4 = 3$



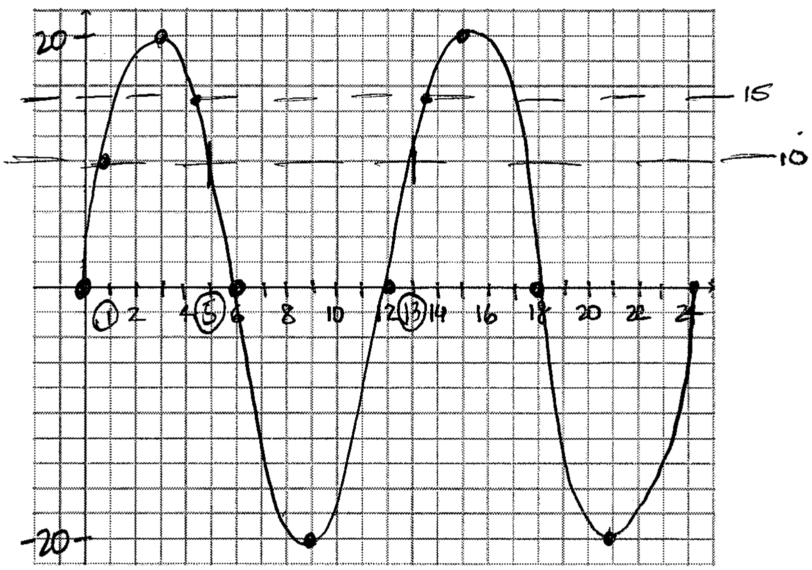
6 hours

6am - 12pm

t = hours since midnight



3. Use the given function and your answers to questions 1 and 2 to sketch the graph of two complete tide cycles. Label and scale the axes.



$10 = 20\sin\frac{\pi}{6}t$

$\frac{1}{2} = \sin\frac{\pi}{6}t$

$\frac{1}{2} = \sin u$

$\frac{\pi}{6}t = \frac{\pi}{6} + 2\pi n$

$\boxed{t = 1 + 12n}$

$\frac{\pi}{6}t = \frac{5\pi}{6} + 2\pi n$

$\boxed{t = 5 + 12n}$

4. Suppose you want to build your castle 10 feet below the average tide line to take advantage of the damp sand. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct? Use your graph to estimate this amount of time and then use algebra & the inverse sine function to find the exact answer.

8 hours

use $t = 1 + 12n$ so 1, 5, 13, 17

$t = 5 + 12n$

furthest time

5. Suppose you want to build your castle 15 feet above the average tide line to give you more time to admire your work. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct? Use your graph to estimate this amount of time and then use algebra & the inverse sine function to find the exact answer.

(using graph) approx 9 hours

$15 = 20\sin\frac{\pi}{6}t$

$\frac{3}{4} = \sin\frac{\pi}{6}t$

(*need calc)

$\frac{6}{\pi})\frac{\pi}{6}t = 0.84806(\frac{6}{\pi}$

$t = 1.6197 + 12n$

$\frac{6}{\pi})\frac{\pi}{6}t = 2.2935(\frac{6}{\pi}$

$t = 4.3803 + 12n$

(using algebra & calculator)

$1.62, \underbrace{4.38, 13.62}, 16.38$

9.24 hrs

6. Suppose you decide you only need two hours to build and admire your castle. What is the lowest point on the beach where you can build the castle?

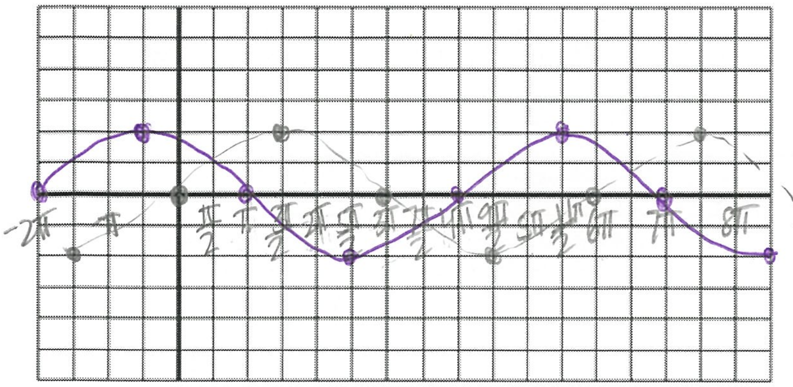
Approx 18 ft below average tide

4.9H Warm Up

Graphing Trig Functions where the Period is rational

$2\sin\left(\frac{1}{3}(x+2\pi)\right)$

1. $y = 2\sin\left(\frac{1}{3}x + \frac{2\pi}{3}\right)$

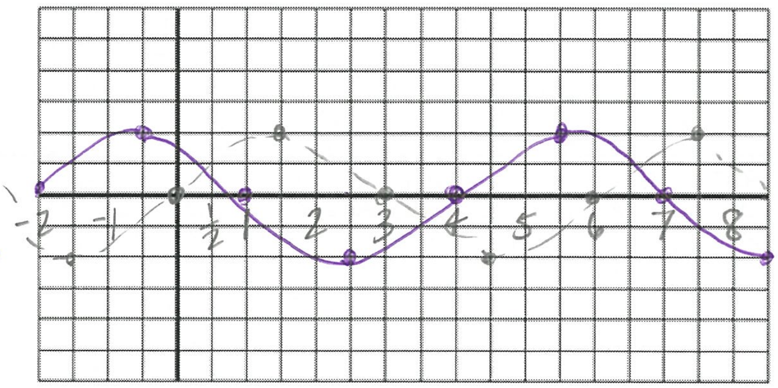


amp = 2
midline = 0
Phase shift = left 2π

period = $2\pi \cdot 3 = 6\pi$
scale: $\frac{6\pi}{4} = \frac{3\pi}{2}$

$2\sin\frac{\pi}{3}(x+2)$

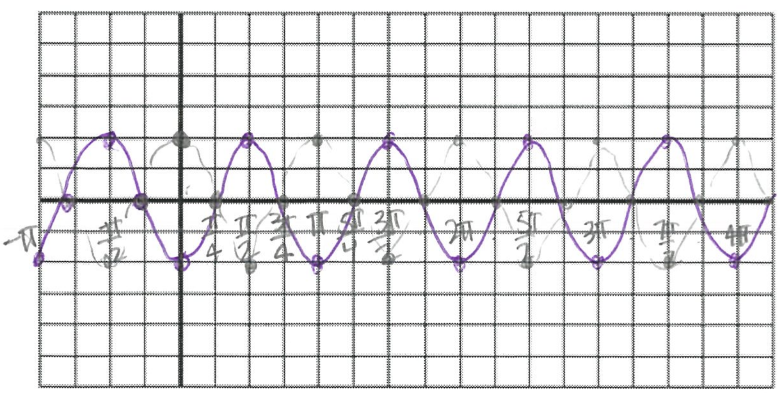
2. $y = 2\sin\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right)$



a = 2
m = 0
PS = 2 left
$p = 2\pi \cdot \frac{3}{\pi} = 6$

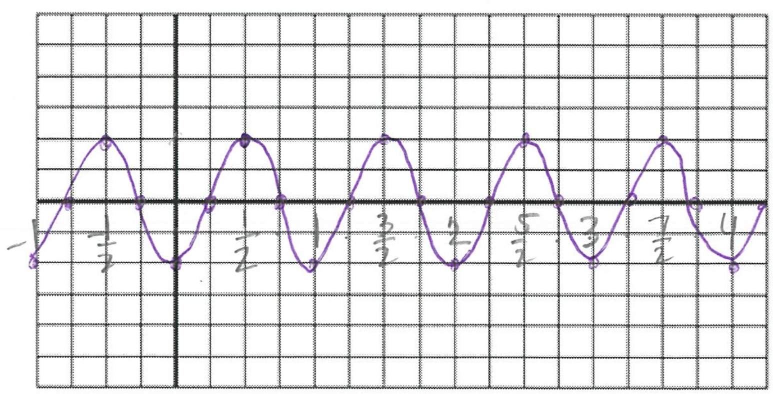
$\frac{6}{4} = \frac{3}{2}$

3. $y = 2\cos(2x + \pi)$ $= 2\cos\left(2\left(x + \frac{\pi}{2}\right)\right)$



period = $\frac{2\pi}{2} = \pi$ scale: $\frac{\pi}{4}$

4. $y = 2\cos(2\pi x + \pi)$ $y = 2\cos\left(2\pi\left(x + \frac{1}{2}\right)\right)$



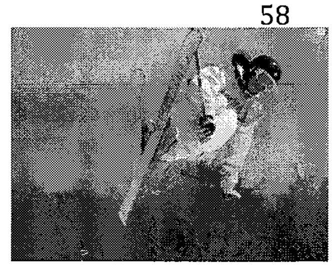
$\frac{2\pi}{2\pi} = 1$

scale $\frac{1}{4}$

5. Why does the period become rational instead of irrational *(in terms of pi)* on problems 2 and 4?

4.9H Off on a Tangent

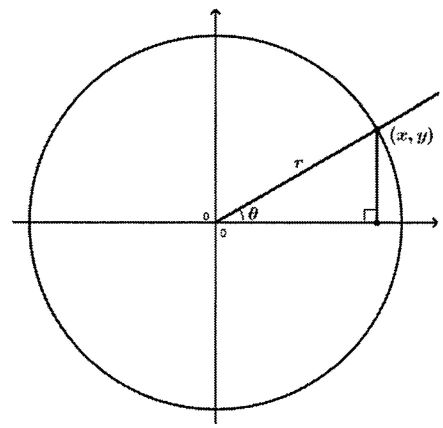
A Develop and Solidify Understanding Task



Recall that the right triangle definition of the tangent ratio is:

$$\tan\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta}$$

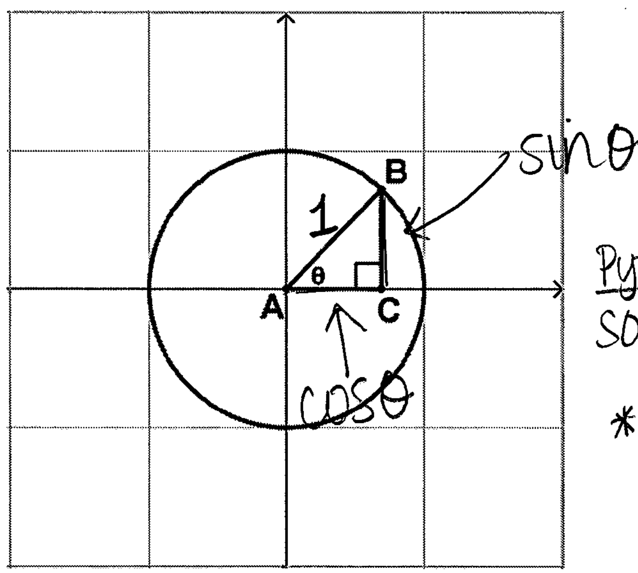
1. Revise the right triangle definition of tangent to find the tangent of **any angle of rotation** drawn in standard position on the unit circle. Explain why your definition is reasonable.



$\tan\theta = \dfrac{y}{x}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

2. We have observed that on the unit circle the value of sine and cosine can be represented with the length of a line segment. Indicate on the following diagram which segment's length represents the value of $\sin\theta$ and which represents the value of $\cos\theta$ for the given angle θ.

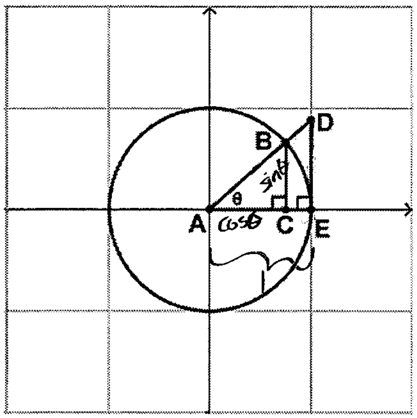


Pythagorean Thr

So... $\sin^2\theta + \cos^2\theta = 1$

* This is where the identity comes from!

3. There is also a line segment that can be defined on the unit circle so that its length represents the value of $\tan\theta$. Consider the length of $\overline{DE}$ in the unit circle diagram below. Note that ΔADE and ΔABC are similar right triangles. Write a convincing argument explaining why the length of $\overline{DE}$ is equivalent to the value of $\tan\theta$ for the given angle, θ.



$\tan\theta = \frac{\overline{DE}}{1}$

so $\tan\theta = DE$

since $\tan\theta = \frac{opp}{adj}$

and we are on the unit circle, the adj = 1

so $\tan\theta = DE$.

4. Extend your thinking about $y = \tan\theta$ by considering the length of $\overline{DE}$ as θ rotates through positive angles from 0 radians to 2π radians. Using your unit circle diagrams from the task, *Water Wheels and the Unit Circle*, give exact values for the following trigonometric expressions:

a. $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$	b. $\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$	c. $\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3}$
d. $\tan\frac{\pi}{4} = 1$	e. $\tan\frac{3\pi}{4} = -1$	f. $\tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3}$
g. $\tan\frac{\pi}{2} =$ undefined	h. $\tan\pi = 0$	i. $\tan\frac{7\pi}{3} = \sqrt{3}$
j. $\tan-\frac{\pi}{3} = -\sqrt{3}$	k. $\tan\frac{3\pi}{2} =$ undefined	l. $\tan-\frac{\pi}{4} = -1$

5. On the coordinate axes below, sketch the graph of $y = \tan\theta$ by considering the length of $\overline{DE}$ as θ rotates through angles from 0 radians to 2π radians. Explain any interesting features you notice in your graph.

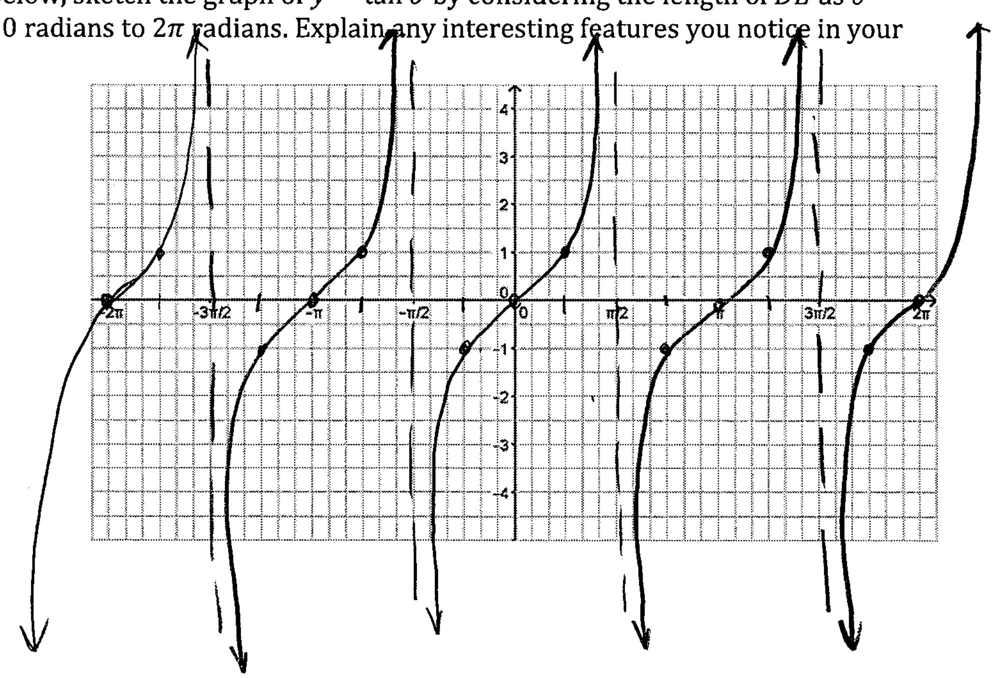
x-intercepts: $0 + \pi n$

Period: π

Asymptotes: $\frac{\pi}{2} + \pi n$

Domain: $x \neq \frac{\pi}{2} + \pi n$

Range: $(-\infty, \infty)$

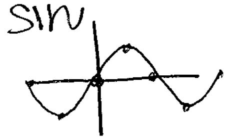


Based on these definitions and your work in this module, determine how to classify each of the following trigonometric functions.

- A function $f(x)$ is classified as an **odd function** if $f(-\theta) = -f(\theta)$.
- A function $f(x)$ is classified as an **even function** if $f(-\theta) = f(\theta)$.

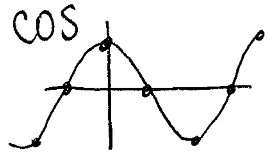
a. The function $y = \sin x$ would be classified as an [**odd function, even function, neither an even or odd function**]. Give evidence for your response.

sin



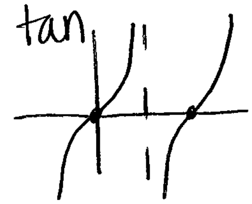
ODD

cos



EVEN

tan



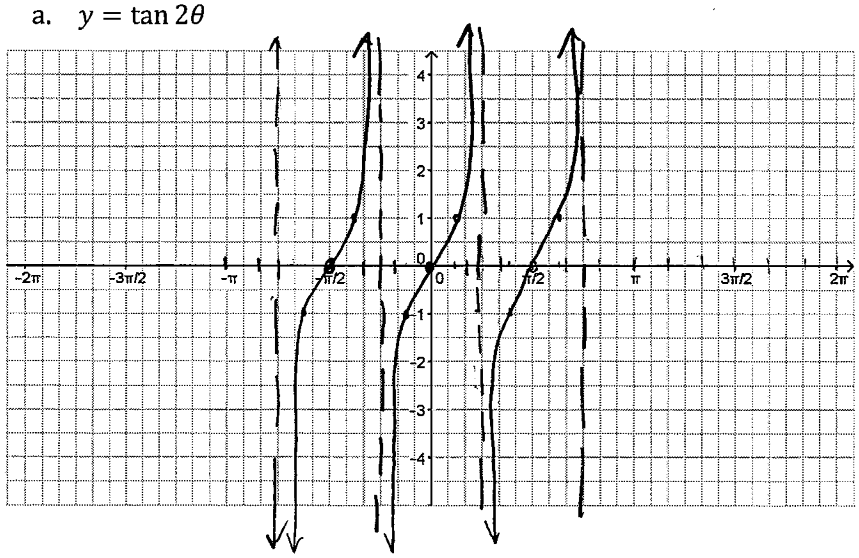
ODD

b. The function $y = \cos x$ would be classified as an [**odd function, even function, neither an even or odd function**]. Give evidence for your response.

c. The function $y = \tan x$ would be classified as an [**odd function, even function, neither an even or odd function**]. Give evidence for your response.

7. Graph each tangent function. Be sure to identify the locations of the asymptotes.

a. $y = \tan 2\theta$



Period: $\frac{\pi}{2}$ scale $\frac{\pi}{8}$

Phase Shift: none

Vertical Shift: none

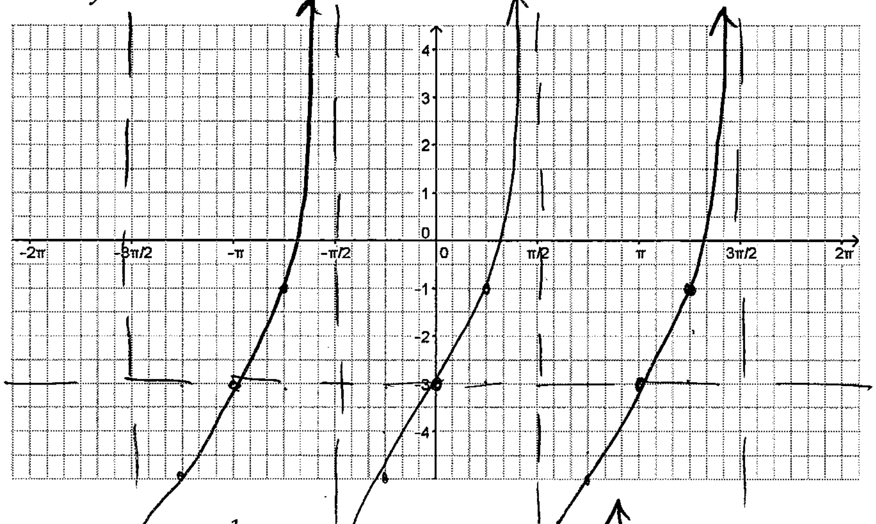
x-intercepts: $0 + \frac{\pi}{2}n$

Asymptotes: $\frac{\pi}{4} + \frac{\pi}{2}n$

Domain:

Range:

b. $y = 2\tan\theta - 3$



Period: π

Phase Shift: none

Vertical Shift: -3

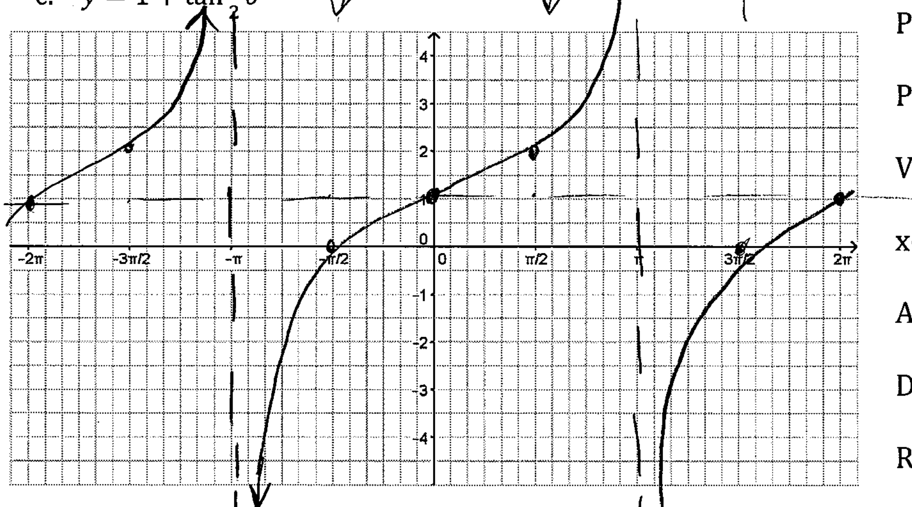
x-intercepts:

Asymptotes:

Domain:

Range:

c. $y = 1 + \tan\frac{1}{2}\theta$



Period: $\frac{\pi}{\frac{1}{2}} = 2\pi$ $\quad \frac{2\pi}{2} = \pi$

Phase Shift:

Vertical Shift:

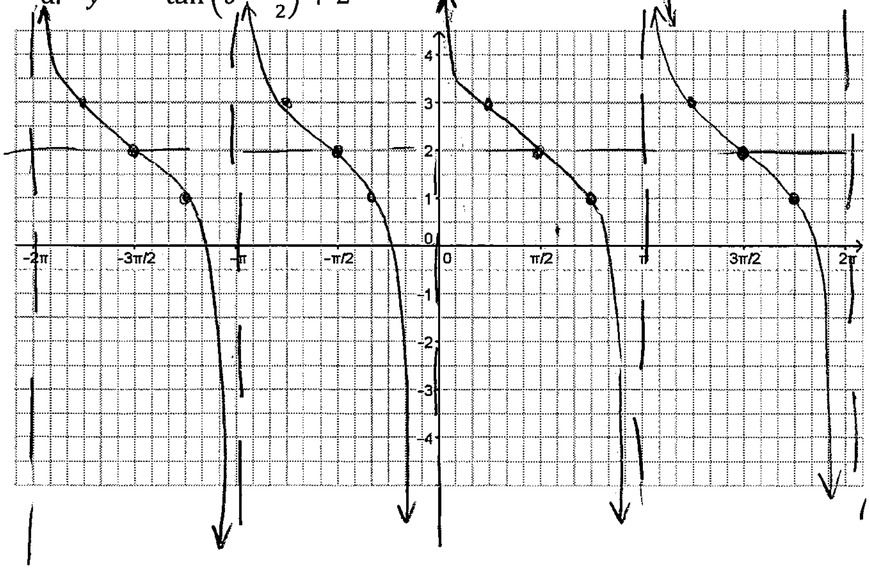
x-intercepts:

Asymptotes:

Domain:

Range:

d. $y = -\tan\left(\theta - \frac{\pi}{2}\right) + 2$



Period:

Phase Shift: $\frac{\pi}{2}$ right

Vertical Shift:

x-intercepts:

Asymptotes:

Domain:

Range:

4.10H Warm Up

Reciprocal Trigonometric Values

Use the triangle below to identify the following trigonometric ratios. Be sure to give exact values.

1. $\sin Q = \frac{4}{4\sqrt{10}} = \frac{\sqrt{10}}{10}$

2. $\csc Q = \frac{4\sqrt{10}}{4} = \sqrt{10}$

3. $\cos Q = \frac{12}{4\sqrt{10}} = \frac{3\sqrt{10}}{10}$

4. $\sec Q = \frac{4\sqrt{10}}{12} = \frac{\sqrt{10}}{3}$

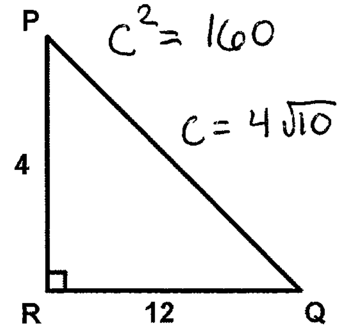
5. $\tan Q = \frac{4}{12} = \frac{1}{3}$

6. $\cot Q = \frac{12}{4} = 3$

$c^2 = 4^2 + 12^2$

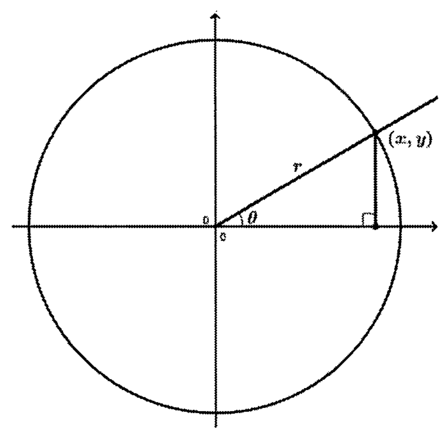
$c^2 = 160$

$c = 4\sqrt{10}$



Recall how we extended the definitions of sine, cosine, and tangent for all angles of rotation:

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$



7. How might the definitions of cosecant, secant, and cotangent be extended to all angles of rotation?

   $\csc\theta = \frac{r}{y}$, $y \neq 0$

   $\sec\theta = \frac{r}{x}$, $x \neq 0$

   $\cot\theta = \frac{x}{y}$, $y \neq 0$

4.10H Reciprocating the Graphs

A Develop and Solidify Understanding Task

Carlos and Clarita were graphing trigonometric functions on their math homework and were wondering what the graphs of the reciprocal functions would look like.

Carlos stated: "Since $\sin\theta$ and $\cos\theta$ both have maximum and minimum values of 1 and -1, I think these will be the maximum and minimum values for $\csc\theta$ and $\sec\theta$ as well."

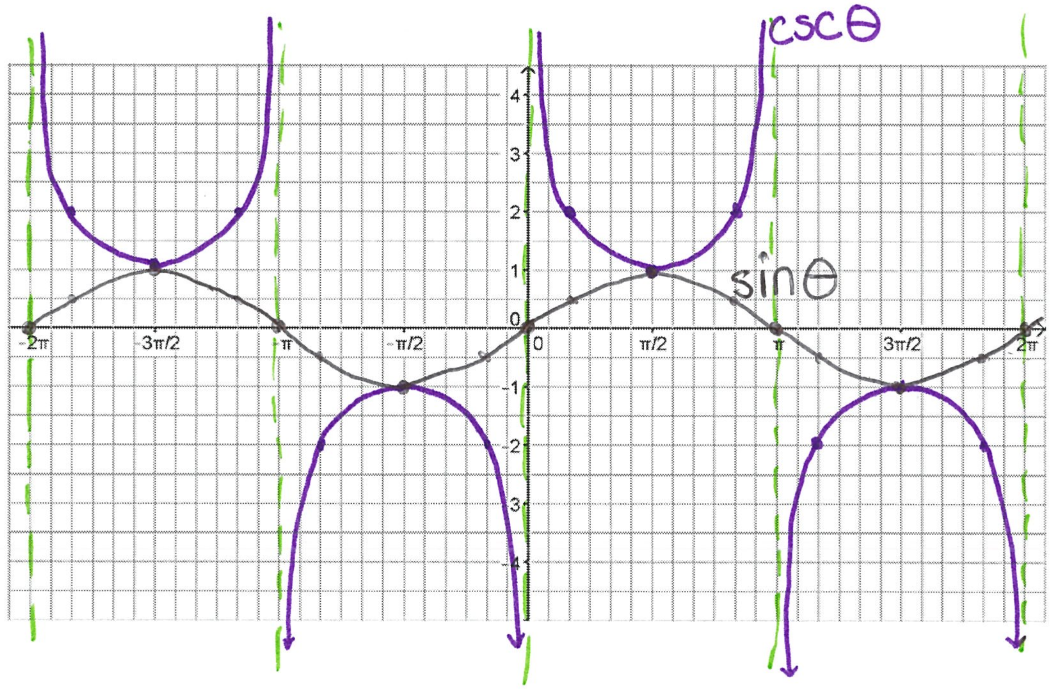
Clarita disagreed: "No way! Think about the values between the zeros and the maximum or between the zeros and the minimum. What happens if you take their reciprocals? The numbers become really large."

1. Who do you agree with? Explain.

   Clarita, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so $\csc\left(\frac{\pi}{6}\right) = 2$ the numbers between $0 < y < 1$ will become larger.

2. Complete the table of values for angles in quadrant I for $y = \sin\theta$ and $y = \csc\theta$. Graph these two functions in different colors on the coordinate plane. Use symmetry to complete the sketch of each function for angles on a domain of $[-2\pi, 2\pi]$. Reminder: $\csc\theta = \frac{1}{\sin\theta}$.

θ	$\sin\theta$	$\csc\theta$
0	0	undef
$\frac{\pi}{6}$	$\frac{1}{2}$	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	1



3. What happens to the graph of $y = \csc\theta$ when the graph of $y = \sin\theta$ crosses the x-axis?

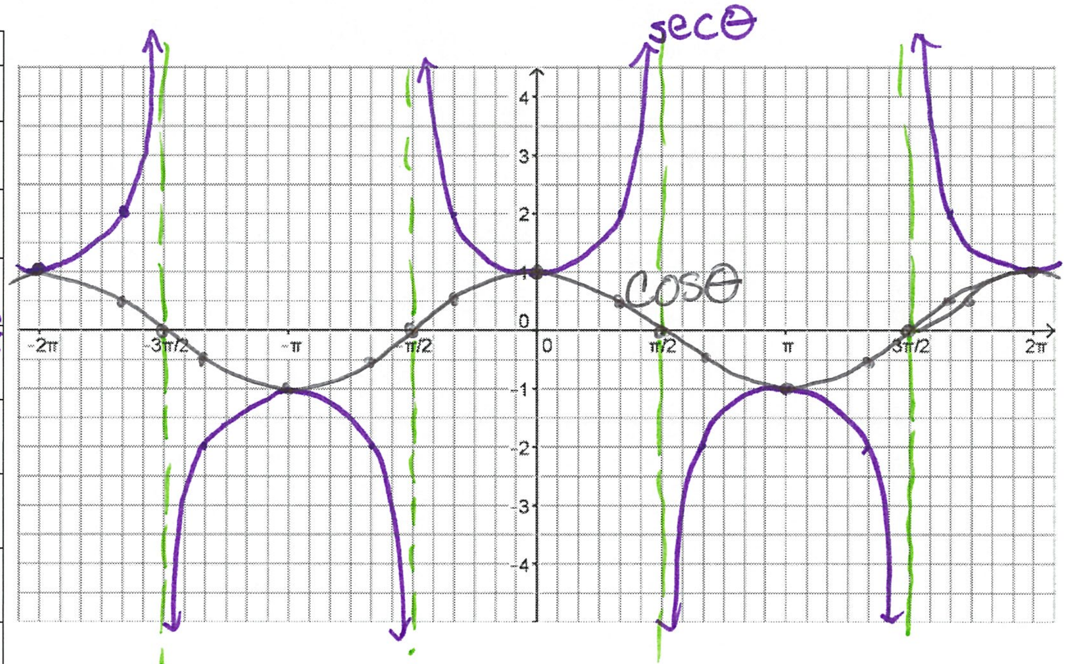
   $y = \csc\theta$ has vertical asymptotes when $\sin\theta = 0$ at $x = 0 + \pi n$

4. Explain why $y = \csc\theta$ has the shape that appears in your graph in question 2.

   As the value of $\sin\theta$ approaches 0, it reciprocal ($\csc\theta$) will approach $\pm\infty$.

5. Complete the table of values for angles in quadrant I for $y = \cos\theta$ and $y = \sec\theta$. Graph these two functions in different colors on the coordinate plane. Use symmetry to complete the sketch of each function for angles on a domain of $[-2\pi, 2\pi]$. Reminder: $\sec\theta = \frac{1}{\cos\theta}$.

θ	$\cos\theta$	$\sec\theta$
0	1	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	2
$\frac{\pi}{2}$	0	undef



6. What happens to the graph of $y = \sec\theta$ when the graph of $y = \cos\theta$ crosses the x-axis?

vertical asymptote. at $x = \frac{\pi}{2} + \pi n$

7. Explain why $y = \sec\theta$ has the shape that appears in your graph in question 5.

as $\cos\theta$ approaches 0, $\sec\theta$ approaches $\pm\infty$

when $\cos\theta = \pm 1$, $\sec\theta = \pm 1$

Clarita says: "The graphs of $y = \csc\theta$ and $y = \sec\theta$ look so strange. What would happen if we take the reciprocals of the values of $\tan\theta$?"

Carlos adds: "That graph will look crazy because $y = \tan\theta$ has asymptotes. What is the reciprocal of an asymptote?"

8. Answer Carlos' question: What is the reciprocal of an asymptote? Hint: think about what type of ratio creates an asymptote.

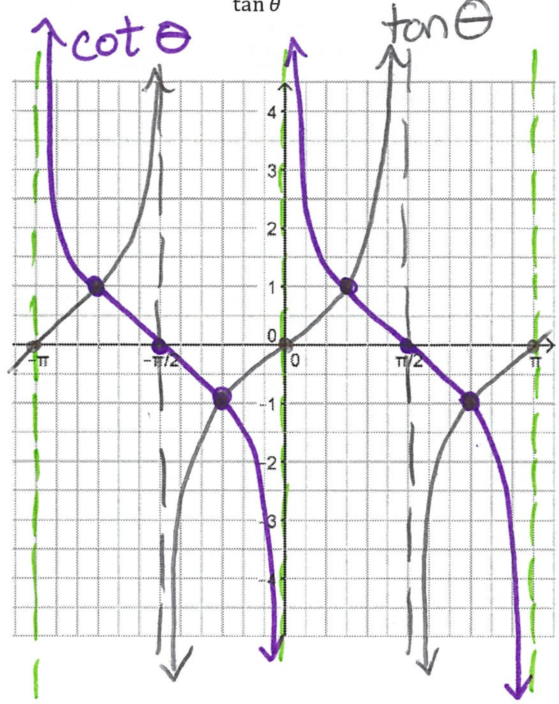
The reciprocal of a vertical asymptote is a x-intercept

if $\tan\theta = \frac{1}{0}$ (asymptote) $\rightarrow \cot(\theta) = \frac{0}{1}$ (x-intercept)

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

9. Complete the table of values for angles in quadrant I for $y = \tan\theta$ and $y = \cot\theta$. Graph these two functions in different colors on the coordinate plane. Use symmetry to complete the sketch of each function for angles on a domain of $[-\pi, \pi]$. Reminder: $\cot\theta = \frac{1}{\tan\theta}$.

θ	$\tan\theta$	$\cot\theta$
0	0	undef
$\frac{\pi}{6}$	$\sqrt{3}/3$	$\sqrt{3}$
$\frac{\pi}{4}$	1	1
$\frac{\pi}{3}$	$\sqrt{3}$	$\sqrt{3}/3$
$\frac{\pi}{2}$	undef	0



10. What happens to the graph of $y = \cot\theta$ when the graph of $y = \tan\theta$ crosses the x-axis?

cot θ has a vertical asymptote when $\tan\theta = 0$

11. What happens to the graph of $y = \cot\theta$ when the graph of $y = \tan\theta$ has an asymptote?

$\cot\theta = 0$ when $\tan\theta$ has a vertical asymptote.

---

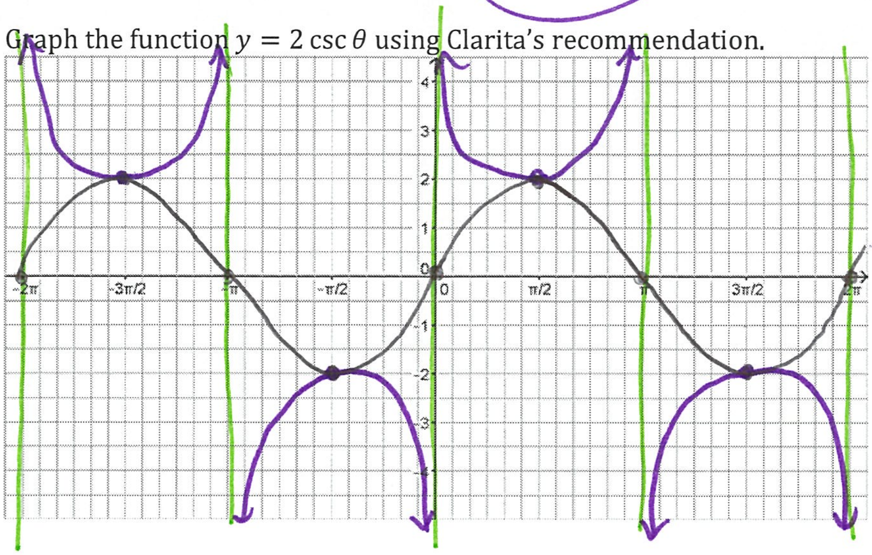
cot θ vertical asymptotes: $x = 0 + \pi n$

tan θ vertical asymptotes: $x = \frac{\pi}{2} + \pi n$

Carlos: "Now that we have seen the basic graphs of $y = \csc\theta$, $y = \sec\theta$, & $y = \cot\theta$, what would happen if we change the numbers in the functions?"

Clarita responds: "Okay. Let's try graphing $y = 2\csc\theta$ by first graphing $y = 2\sin\theta$."

12. Graph the function $y = 2\csc\theta$ using Clarita's recommendation.

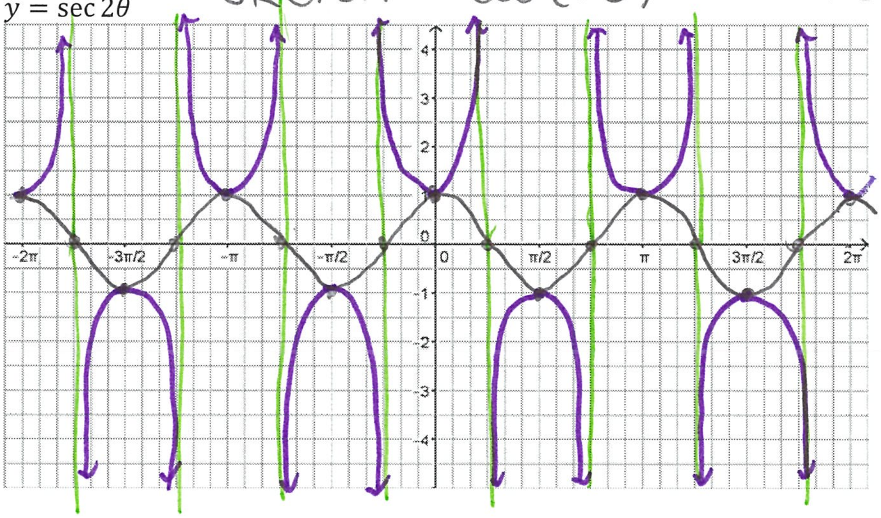


Carlos says: "I'm starting to understand. Let's try a few more."

Graph the following functions by first graphing the associated sine, cosine, or tangent function.

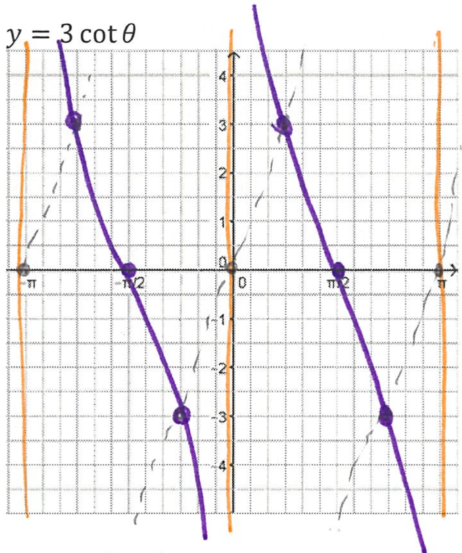
13. $y = \sec 2\theta$ sketch $\cos(2\theta) \longrightarrow$ Period $= \frac{2\pi}{2} = \pi$

scale $= \frac{\pi}{4}$



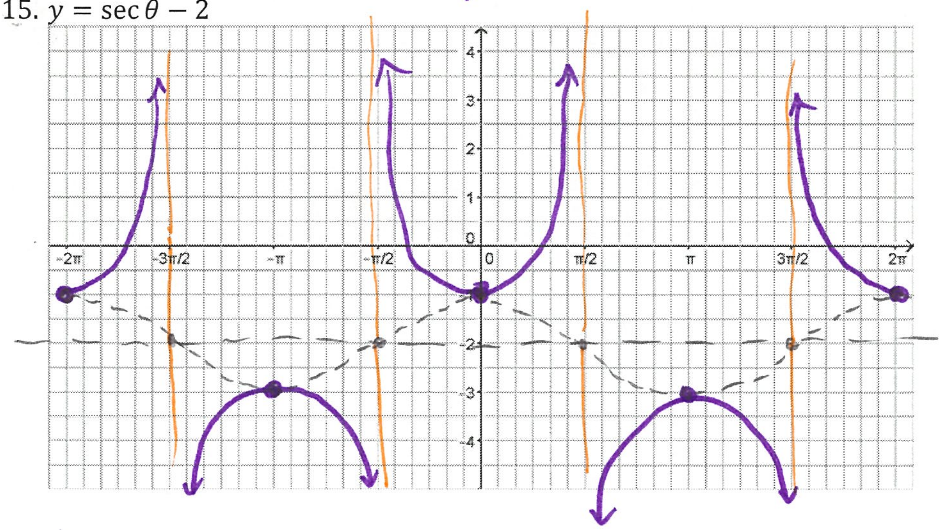
sketch $3\tan\theta$

14. $y = 3\cot\theta$



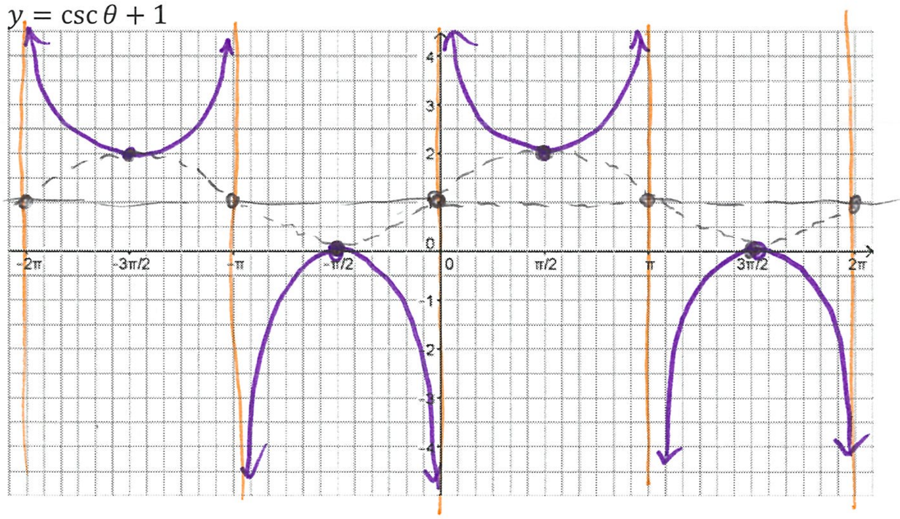
15. $y = \sec\theta - 2$

sketch $\cos\theta - 2$



16. $y = \csc\theta + 1$

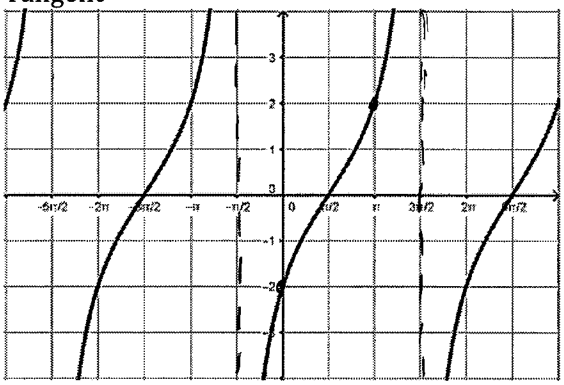
sketch $\sin\theta + 1$



4.11H Warm Up
What's Your Equation?

For each graph below, write the indicated function.

1. Tangent

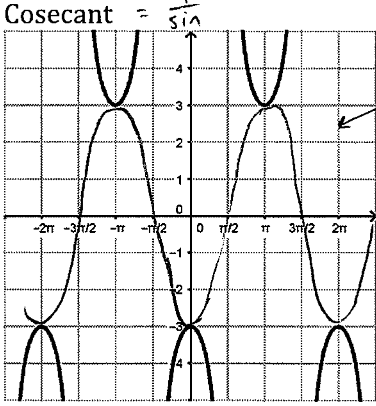


$$\boxed{y = 2\tan\left(\tfrac{1}{2}\left(x - \tfrac{\pi}{2}\right)\right)}$$

$\frac{\pi}{b} = 2\pi$

$b = \frac{1}{2}$

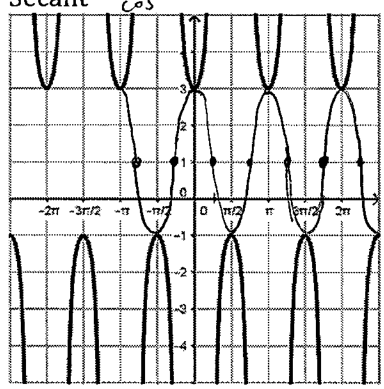
2. Cosecant $= \frac{1}{\sin}$



$y = 3\sin\left(x - \frac{\pi}{2}\right)$

$$\boxed{y = 3\csc\left(x - \tfrac{\pi}{2}\right)}$$

3. Secant $= \frac{1}{\cos}$



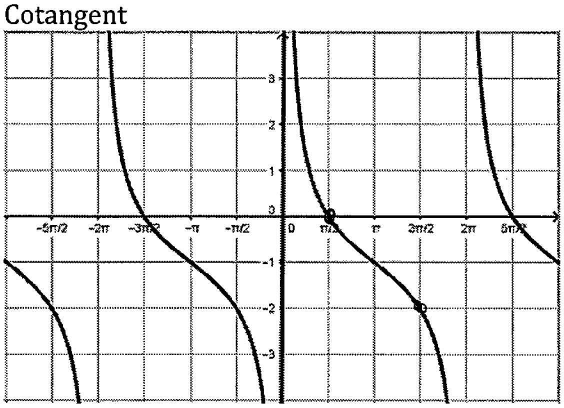
Period $= \pi$

$\frac{2\pi}{b} = \pi \qquad b = 2$

$y = 2\cos 2x + 1$

$$\boxed{y = 2\sec 2x + 1}$$

4. Cotangent



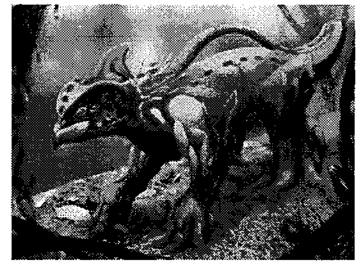
$$\boxed{y = \cot\left(\tfrac{1}{2}x\right) - 1}$$

$P = 2\pi$

$\frac{\pi}{b} = 2\pi$

$b = \frac{1}{2}$

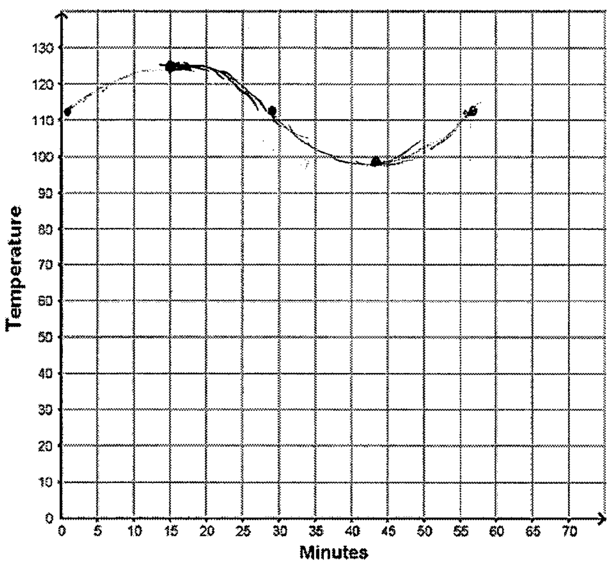
4.11H Modeling and Inverses
A Solidify and Practice Understanding Task



Part I:

1. A team of biologists have discovered a new creature in the rain forest. They note the temperature of the animal appears to vary periodically over time. A maximum temperature of 125° occurs 15 minutes after they start their examination. A minimum temperature of 99° occurs 28 minutes later. The team would like to find a way to predict the animal's temperature over time in minutes.

   a. Create a graph of one full period.



   b. Write an equation of temperature as a function over time in minutes using sine.

$$Y = 13\sin\left(\frac{\pi}{28}(t-1)\right) + 112 \qquad \frac{2\pi}{56} = \frac{\pi}{28}$$

   c. Write an equation of temperature as a function over time in minutes using cosine.

$$y = 13\cos\left(\frac{\pi}{28}(t-15)\right) + 112$$

   d. Use your sine or cosine equations from parts a and b to write an equation to represent ***all*** the times when the temperature will be 108°.

$$13\sin\left(\frac{\pi}{28}(t-1)\right) + 112 = 108$$

$$13\sin\left(\frac{\pi}{28}(t-1)\right) = -4$$

$$\sin\left(\frac{\pi}{28}(t-1)\right) = \frac{-4}{13}$$

$$\frac{\pi}{28}(t-1) = \sin^{-1}\left(-\frac{4}{13}\right)$$

$$t - 1 = \frac{28}{\pi}\sin^{-1}\left(-\frac{4}{13}\right)$$

$$t = \frac{28}{\pi}\sin^{-1}\left(-\frac{4}{13}\right) + 1$$

   e. Give at least two times when the temperature will be 108°.

$t \approx -1.78758..$ $\quad +56 \quad$ $\boxed{t \approx 54.212411}$

or $t \approx 29 + 2.78758 = \boxed{31.78758}$

symmetry

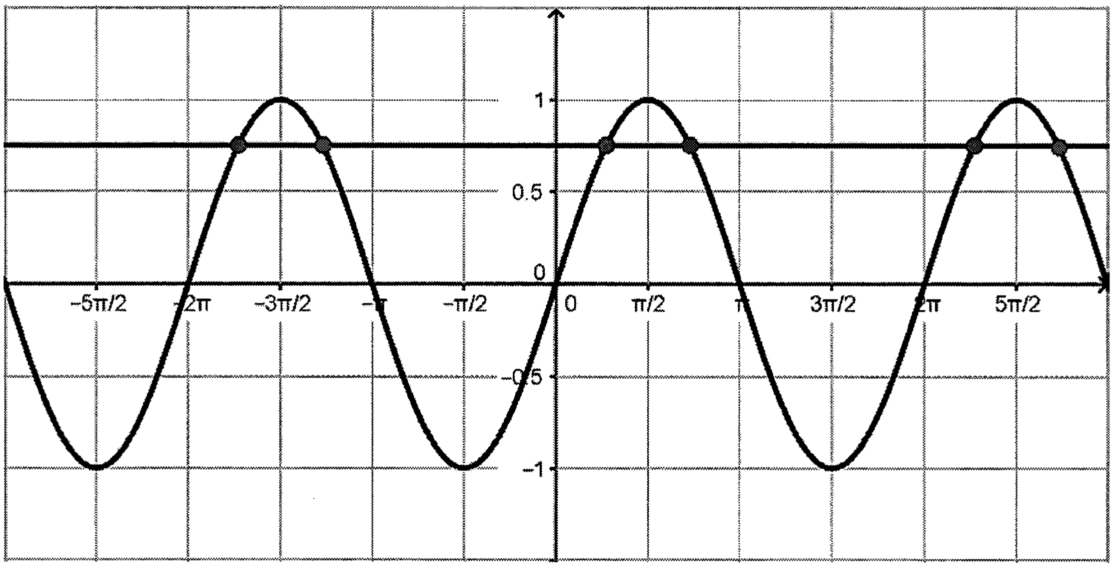
Part II:

Perhaps you thought about the unit circle or used a calculator to answer the previous question. For periodic functions, there are many answers to this type of question. Therefore, this question, by itself, does not define an inverse trigonometric function.

Suppose we have a simplified equation:

$$\sin\theta = 0.75$$

Using your calculator, $\sin^{-1} 0.75 = 0.848$ radians. However, the following graph indicates other values of θ for which $\sin\theta = 0.75$



1. Without tracing the graph or using any other calculator analysis tools, use the fact that $\sin^{-1} 0.75 = 0.848$ radians to find at least three other angles, θ, where $\sin\theta = 0.75$. Each of these angles shows up as a point of intersection between the sine curve and the line $y = 0.75$ in the graph shown above.

$\theta = \pi - .848$

$\theta = .848 + 2\pi$

$\theta = -\pi - .848$

Your calculator has been programmed to use the following definition for the inverse sine function, so that each time we find $\sin^{-1}$ of a number, we will get **exactly one** solution.

***Definition of the Inverse Sine Function*:**

$y = \sin^{-1}x$ means, "find the angle *y, on the interval* $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, such that $\sin y = x$."

2. Based on the graph of the sine function, explain why defining the inverse trigonometric function in this way guarantees that it will have a single, unique output.

   By restricting the inverse's range, it is now a function!

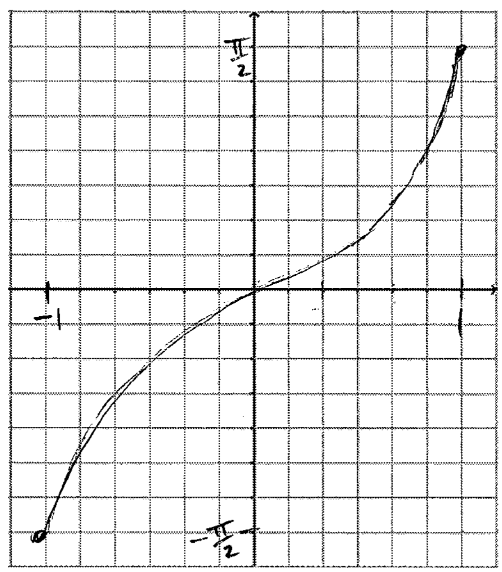
3. Based on this definition, what is the **domain** of this inverse trigonometric function?

   $[-1, 1]$

4. Based on this definition, what is the **range** of this inverse trigonometric function?

   $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

5. Sketch a graph of the inverse sine function. Label your axes.



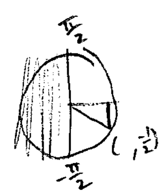
6a. Solve the equation.

$\sin\theta = -\frac{1}{2}$.

$\theta = \frac{7}{6}\pi + 2\pi n$

$\theta = \frac{11}{6}\pi + 2\pi n$

6b. Evaluate the expression, $\sin^{-1}\left(-\frac{1}{2}\right) = \frac{-\pi}{6}$ or $-30°$



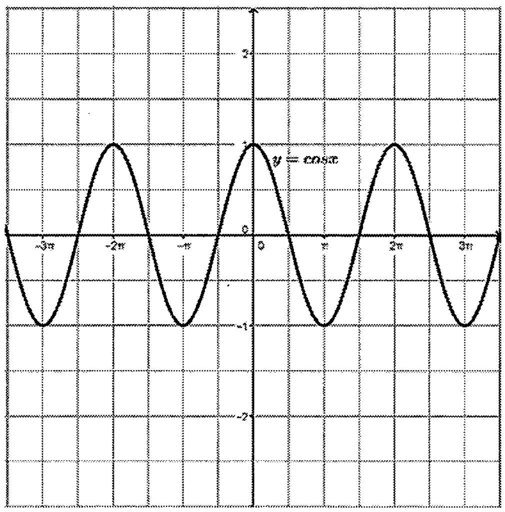
6c. How are the answers to 6a and 6b different?

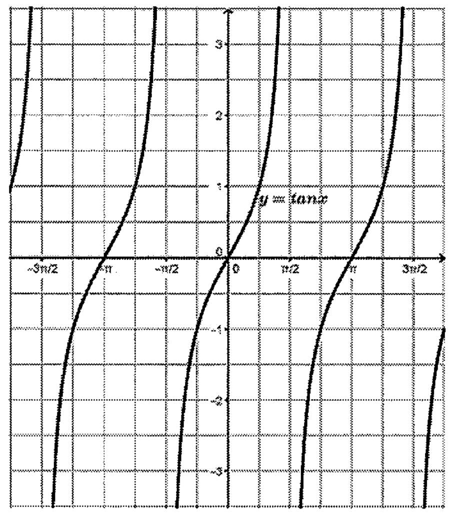
a) We are finding all θ that makes the equation true.

b) We start with $\sin^{-1}(-\frac{1}{2})$ which has a restricted range from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Examine the graphs of the cosine and tangent functions below. How would you restrict the domains of these trigonometric functions so that the inverse cosine function and the inverse tangent function can be constructed?

Complete the definitions of the inverse cosine function and the inverse tangent function below. State the domain and range of each function, and sketch its graph.



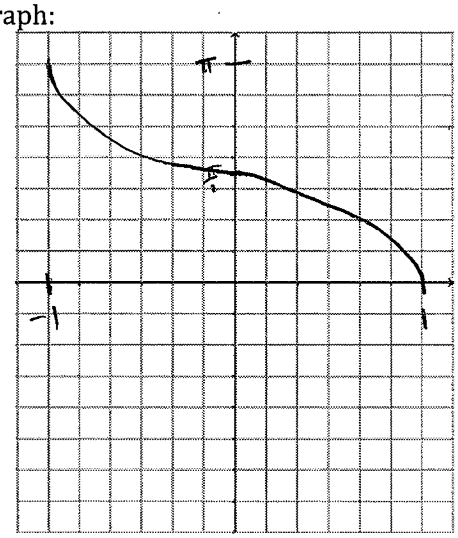


Definition of the Inverse Cosine Function:

Domain: $[-1, 1]$

Range: $[0, \pi]$

Graph:



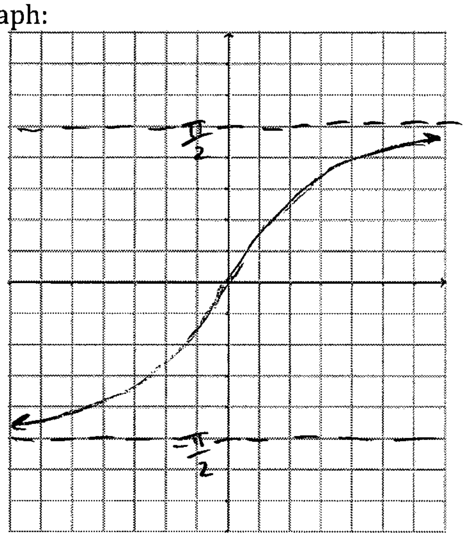
7. Find: $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\quad \cos^{-1}(-1)$ $\quad \cos^{-1}\left(\frac{1}{2}\right)$

$\frac{5\pi}{6}$ $\quad \pi$ $\quad \frac{\pi}{3}$

<u>Definition of the Inverse Tangent Function</u>:

Domain: $(-\infty, \infty)$ Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Graph:



8. Find: $\tan^{-1}(-\sqrt{3})$ $\quad -\frac{\pi}{3}$ $\qquad \tan^{-1}(-1)$ $\quad -\frac{\pi}{4}$ $\qquad \tan^{-1}(1)$ $\quad \frac{\pi}{4}$

Another way to write the inverse trigonometric functions is by using the following notation:

$$\sin^{-1}(x) = \arcsin(x)$$

$$\cos^{-1}(x) = \arccos(x)$$

$$\tan^{-1}(x) = \arctan(x)$$

Find the exact values of the following:

9. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ $\quad \frac{3\pi}{4}$

10. $\arctan\left(\frac{\sqrt{3}}{3}\right)$ $\quad \frac{\pi}{6}$

11. $\arcsin\left(\frac{1}{2}\right)$ $\quad \frac{\pi}{6}$

Find the values of the following correct to four decimal places:

12. $\arctan(-\sqrt{473})$ $\quad -1.5248$

13. $\arcsin(-0.625)$ $\quad -.6751$

14. $\arccos\left(\frac{\sqrt{7}}{10}\right)$ $\quad 1.3030$

Practice:

Find the exact value without a calculator.

1. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

$\cos\left(\frac{\pi}{6}\right)$

$\boxed{\frac{\sqrt{3}}{2}}$

2. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

$\sin\left(\frac{\pi}{4}\right)$

$\boxed{\frac{\sqrt{2}}{2}}$

3. $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\boxed{\frac{\pi}{6}}$

4. $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\boxed{\frac{\pi}{3}}$

5. $\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right)$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\boxed{\frac{\pi}{3}}$

6. $\cos^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right)$

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\boxed{\frac{5\pi}{6}}$

7. $\tan^{-1}(\sin(\pi))$

$\tan^{-1}(0)$

$\boxed{0}$

8. $\sin\left(\arctan(\sqrt{3})\right)$

$\sin\left(\frac{\pi}{3}\right)$

$\boxed{\frac{\sqrt{3}}{2}}$

9. $\cos\left(\arctan(-1)\right)$

$\cos\left(-\frac{\pi}{4}\right)$

$\boxed{\frac{\sqrt{2}}{2}}$

10. $\arccos\left(\sin\left(-\frac{7\pi}{4}\right)\right)$

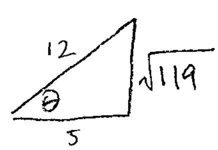
$\arccos\left(\frac{\sqrt{2}}{2}\right)$

$\boxed{\frac{\pi}{4}}$

11. $\sin\left(\arccos\left(\frac{5}{12}\right)\right)$

$\sin(\theta)$

$\boxed{\frac{\sqrt{119}}{12}}$



12. $\sin\left(\arctan\left(-\frac{7}{11}\right)\right)$

$\sin(\theta)$

$\boxed{\frac{-7}{\sqrt{170}}}$